# Generalized Feistel Structures Based on Tweakable Block Ciphers

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FSE 2023

Beijing / Kobe

March 24, 2023

# Outline

- Introduction
- Our Contributions
- Security Proofs
- Matching Attacks
- Conclusions

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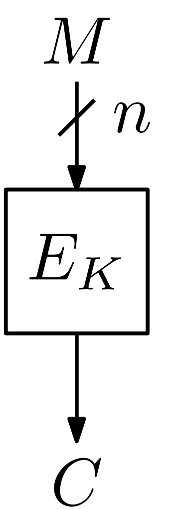
# Block Ciphers

- block cipher (BC)
  - a keyed permutation $E: \mathcal{K} \times \{0,1\}^n \rightarrow \{0,1\}^n$
  - for any key $K \in \mathcal{K}$, $E_K(\cdot)$ is a permutation over $\{0,1\}^n$
  - $n$ is the block length, $n$-BC

- Construction of a secure block cipher is one of the most important problems in symmetric key cryptography.



# Secure Block Ciphers

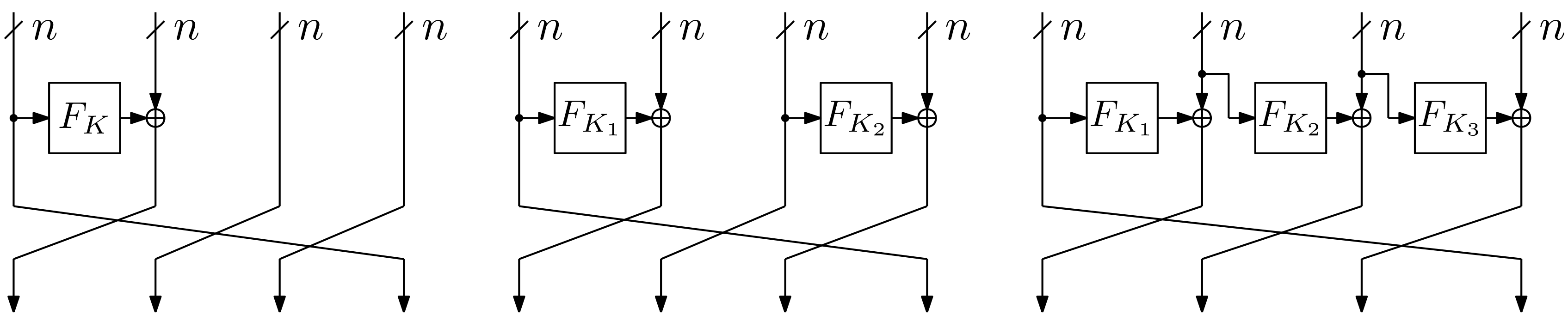
- pseudorandom permutation (PRP) [LR88]
  - real world: $E_K$, $n$-BC
  - ideal world: $\pi$, $n$-bit random permutation
  - $\mathbf{Adv}_E^{\mathrm{prp}}(\mathcal{A}) = \left|\Pr\left[\mathcal{A}^{E_K(\cdot)} = 1\right] - \Pr\left[\mathcal{A}^{\pi(\cdot)} = 1\right]\right|$
- strong pseudorandom permutation (SPRP) [LR88]
  - real world: $(E_K, E_K^{-1})$
  - ideal world: $(\pi, \pi^{-1})$
  - $\mathbf{Adv}_E^{\mathrm{sprp}}(\mathcal{A}) = \left|\Pr\left[\mathcal{A}^{E_K(\cdot), E_K^{-1}(\cdot)} = 1\right] - \Pr\left[\mathcal{A}^{\pi(\cdot), \pi^{-1}(\cdot)} = 1\right]\right|$

- Feistel structure [LR88]
  - 3-round Feistel with $n$-bit pseudorandom functions (PRFs) is a PRP
  - 4-round Feistel with $n$-bit PRFs is an SPRP

# Generalized Feistel Structures

- generalized Feistel structures (GFSs)
  - generalization of Feistel structure
  - unbalanced GFS [SK96],
    type-1, type-2, and type-3 GFSs [ZMI89], …

- type-1, type-2, type-3 GFSs [ZMI89]
  - type-1: $(2d-1)$-round is a PRP
  - type-2: $(d+1)$-round is a PRP, $(d+2)$-round is an SPRP
  - type-3: $(d+1)$-round is a PRP

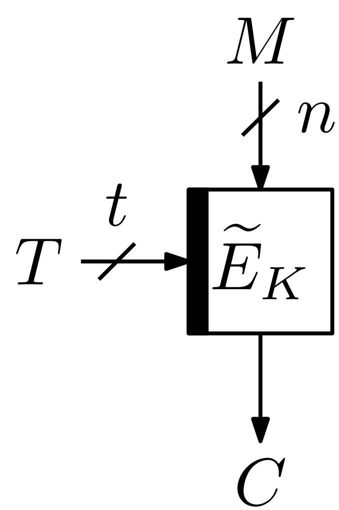



$dn$-bit type-1, type-2, type-3 GFSs $(d = 4)$

# Tweakable Block Ciphers

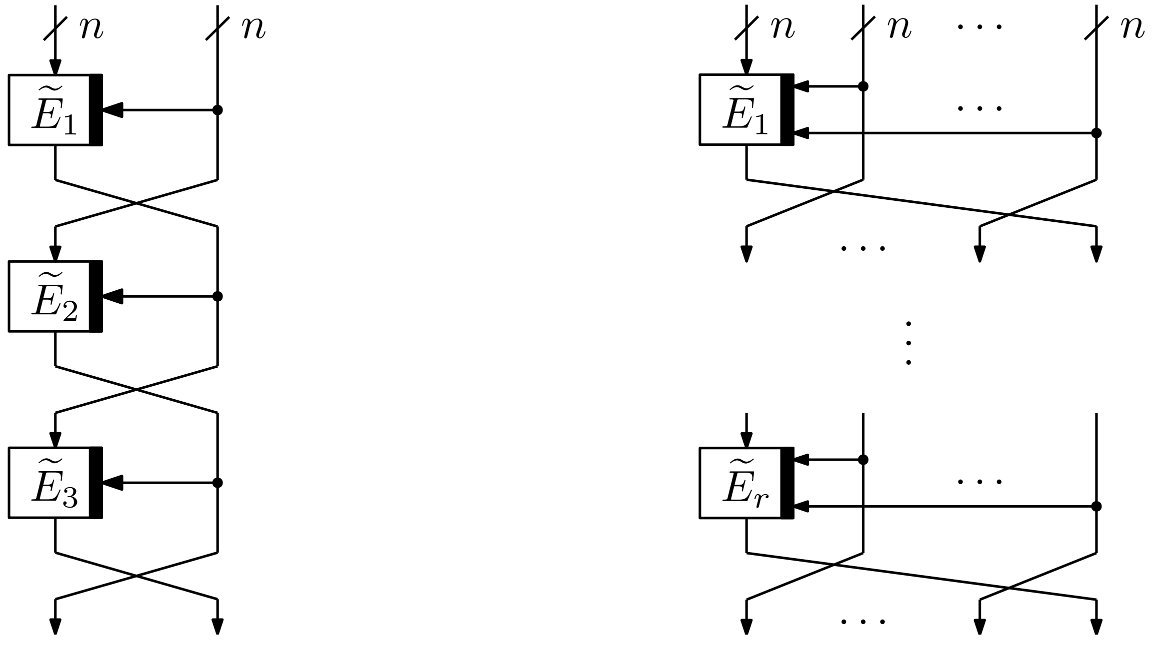
- tweakable block cipher (TBC) [LRW02, LRW11]
  - $\tilde{E}: \mathcal{K} \times \{0,1\}^t \times \{0,1\}^n \rightarrow \{0,1\}^n$
  - $T \in \{0,1\}^t$ is an additional input called a tweak
  - for any key $K \in \mathcal{K}$ and any tweak $T \in \{0,1\}^t$, $\tilde{E}_K(T,\cdot)$ is a permutation over $\{0,1\}^n$
  - $t$-bit tweak and $n$-bit block TBC, $(t,n)$-TBC

- secure TBCs from secure block ciphers [LRW02, LRW11]
- secure block ciphers from secure TBCs [Min09]



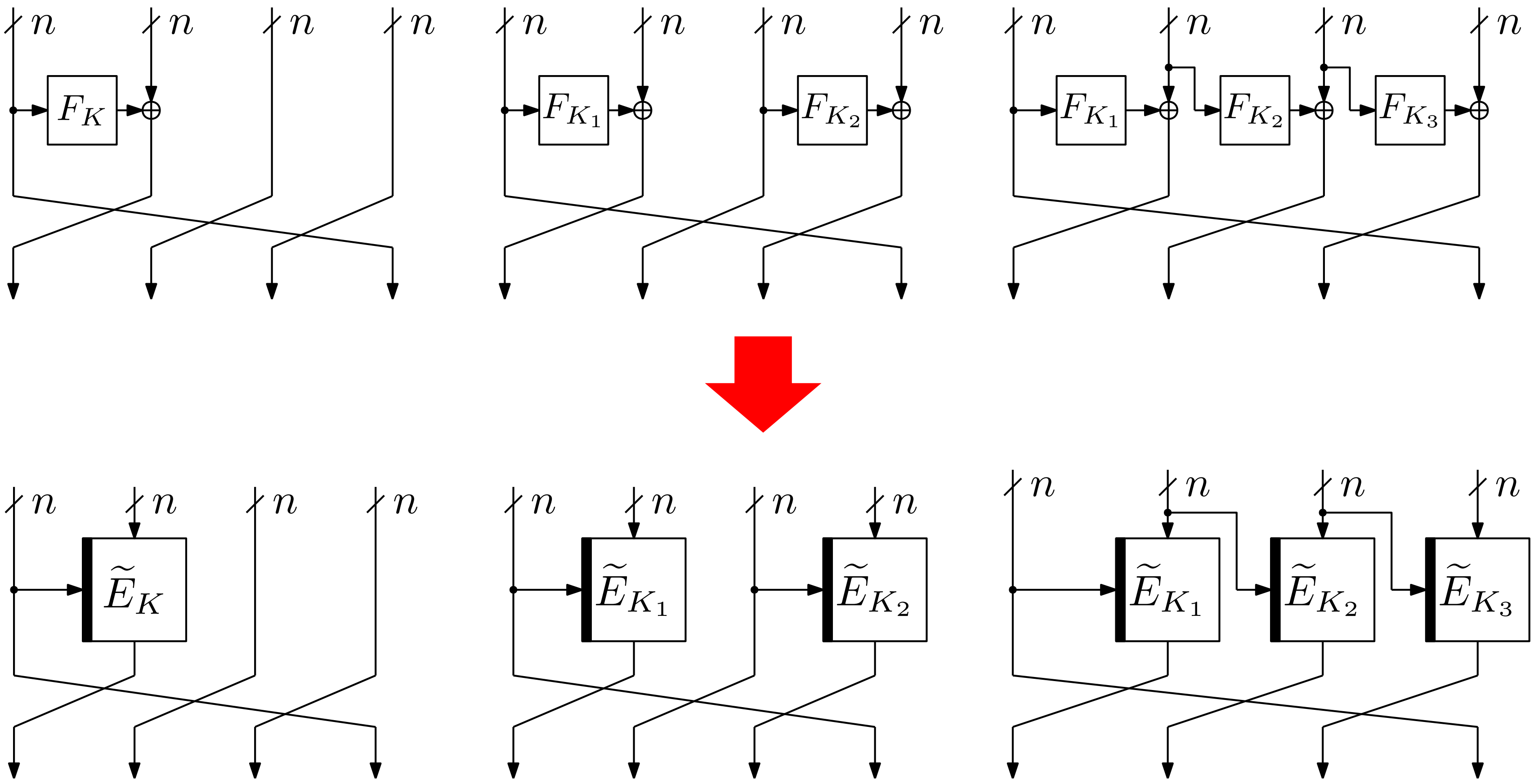
# Secure Block Ciphers from TBCs

- Coron et al. [CDMS10]
  - $2n$-BC from $(n, n)$-TBC
  - Feistel structure
- Minematsu and Nakamichi et al. [Min15,NI19]
  - $dn$-BC from $\left((d-1)n, n\right)$-TBC
  - unbalanced GFS



# GFSs based on TBCs

- Type-1, 2, 3 GFSs based on TBCs can naturally be defined
  - $n$-bit PRF and XOR → $(n, n)$-TBC



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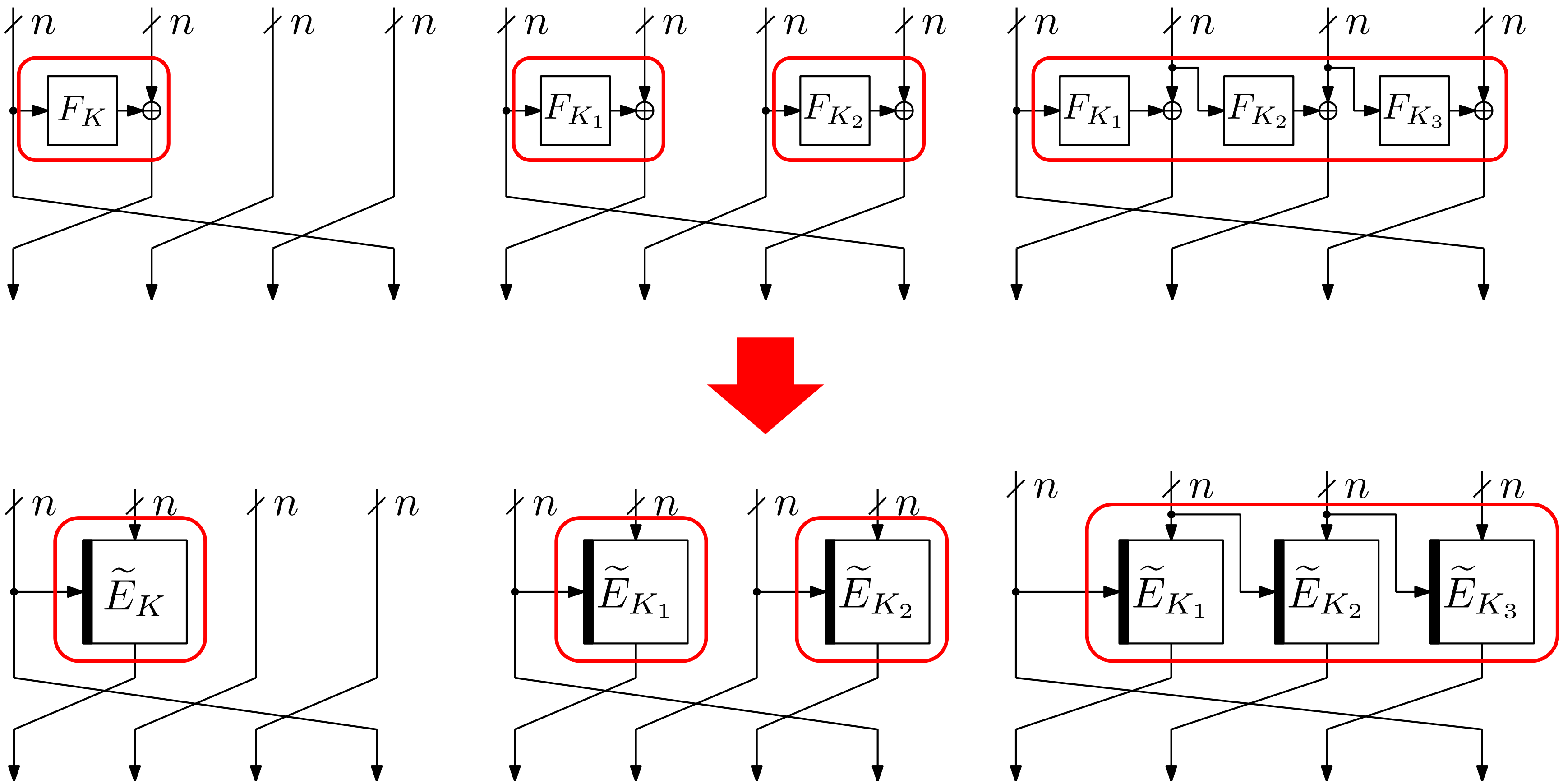

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# Our Contributions

| Model | Prim. | Const. | Security bound | # of rounds | Reference |
|---|---|---|---|---|---|
| PRP | TBC | Type-1 | $O(q^2/2^n)$ | $2d-2$ | This paper |
| | | | $O(q^2/2^{2n})$ | $3d-2$ | |
| SPRP | TBC | Type-1 | $O(q^2/2^n)$ | $d^2-2d+2$ | |
| | | | $O(q^2/2^{2n})$ | $d^2-d+2$ | |
| | | Type-2 | $O(q^2/2^n)$ | $d$ | |
| | | | $O(q^2/2^{2n})$ | $d+2$ | |
| | | Type-3 | $O(q^2/2^n)$ | $d$ | |
| | | | $O(q^2/2^{2n})$ | $d+1$ | |

- these primitives are $(n,n)$-TBCs, the constructions are $dn$-BCs
- $q$ is the number of queries

- We identify the number of rounds needed to achieve birthday-bound security and BBB security (with respect to $n$).
  - BBB: beyond-birthday-bound

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| | | Type-2 | $O(q^2/2^n)$ | $d$ | |
| | | | $O(q^2/2^{2n})$ | $d+2$ | |
| | | Type-3 | $O(q^2/2^n)$ | $d$ | |
| | | | $O(q^2/2^{2n})$ | $d+1$ | |

- For type-1 GFS, we prove PRP and SPRP security separately
  - this construction has different security characteristics depending on the direction of the operation
- For type-2 and type-3 GFSs, we prove SPRP security

# Our Contributions

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| | | | $O(q^2/2^{2n})$ | $d+1$ | |

- We also analyse the optimality of our results with respect to the number of rounds and the attack complexity.
- We note that the constructions we consider in this paper have iterative structures

# Related Works

| Model | Prim. | Const. | Security bound | # of rounds | Reference |
|---|---|---|---|---|---|
| SPRP | PRF | Type-1 | $O\left(\frac{q^{t+1}}{2^{nt}}\right)$ | $(d^2+d-2)t+1$ | [SGW20] |
| | | Type-2 | | $2dt+1$ | |
| | | Type-3 | | $(d+2)t+1$ | |
| SPRP | PRF | Feistel | $O(q^2/2^n)$ | $4$ | [LR88] |
| | TBC | Feistel | $O(q^2/2^{2n})$ | $3$ | [CDMS10] |
| | | | $O\left(\frac{q^{(t+1)/2}}{2^{nt}}\right)$ | $4t+1$ | [SGW20] |

- in the results of [SGW20], $t \geq 1$ is a parameter that specifies the number of rounds
  - proved stronger security bounds than previous results by increasing the number of rounds

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# Coefficient-H Technique

- interpolation probability
  - in the real world: $\Pr[\Theta_{\mathcal{R}} = \theta]$
  - in the ideal world: $\Pr[\Theta_{\mathcal{I}} = \theta]$

- an attainable transcript:
  a transcript $\theta$ that satisfies $\Pr[\Theta_{\mathcal{I}} = \theta] > 0$

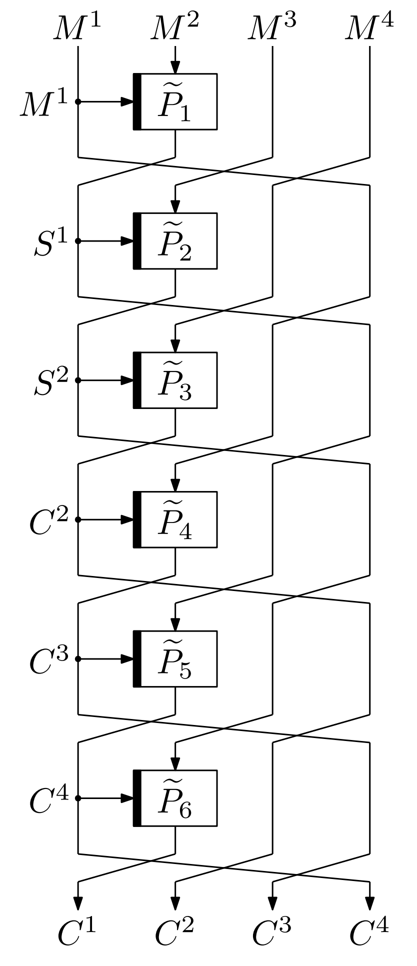
- Coefficient-H technique [Pat08, CS14]
  - partition all the attainable transcripts into $\mathrm{T}_{\mathrm{good}}$ and $\mathrm{T}_{\mathrm{bad}}$
  - assume that there exists $0 \leq \epsilon \leq 1$ such that:
    $$\forall \theta \in \mathrm{T}_{\mathrm{good}}, \qquad \frac{\Pr[\Theta_{\mathcal{R}} = \theta]}{\Pr[\Theta_{\mathcal{I}} = \theta]} \geq 1 - \epsilon$$
  - Then, $\mathbf{Adv}_{E}^{(\mathrm{model})}(\mathcal{A}) \leq \epsilon + \Pr[\Theta_{\mathcal{I}} \in \mathrm{T}_{\mathrm{bad}}]$,
    where $(\mathrm{model}) \in \{\mathrm{prp}, \mathrm{sprp}\}$ depending on the queries

- $\epsilon$ and $\Pr[\Theta_{\mathcal{I}} \in \mathrm{T}_{\mathrm{bad}}]$ depend on the definitions of the oracles and $\mathrm{T}_{\mathrm{bad}}$

# Oracle Definitions

- The real world oracle $\mathcal{R}$ : TBC-based type-1, 2, 3 GFS
  - for each query, $\mathcal{R}$ records all the internal states in $\mathcal{S}$
    $\rightarrow$ adversary $\mathcal{A}$ gets $\mathcal{S}$ after $\mathcal{A}$ makes all the queries

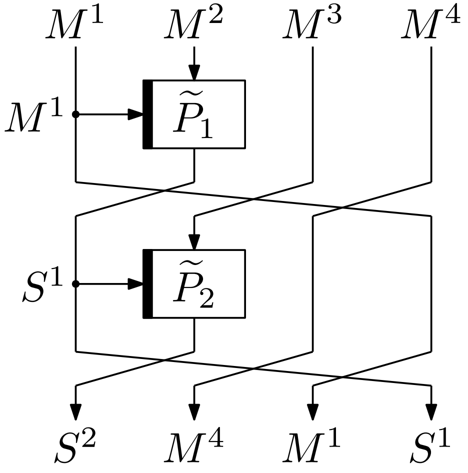
- Example: (PRP proof, birthday-bound)
  Type-1 GFS with $d = 4, r = 2d - 2 = 6$
  - computes the internal states $S^1$ and $S^2$ with $\tilde{P}_1$ and $\tilde{P}_2$
  - computes the ciphertext with $\tilde{P}_3, \ldots, \tilde{P}_6$

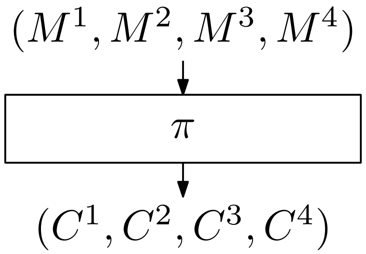


# Oracle Definitions

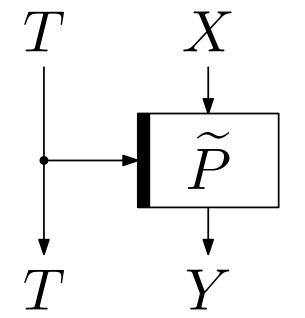
- The ideal world oracle $\mathcal{I}$ : $dn$-bit random permutation $\pi$
  - for each query, $\mathcal{I}$ uses dummy TBCs to compute dummy internal states
    (same probability distribution as in the real world)
  - adversary $\mathcal{A}$ gets $\mathcal{S}$ after $\mathcal{A}$ makes all the queries

- Example: (PRP proof, birthday-bound)
  Type-1 GFS with $d = 4, r = 2d - 2 = 6$
  - computes the internal states $S^1$ and $S^2$ with $\tilde{P}_1$ and $\tilde{P}_2$
  - computes the ciphertext with $\pi$



# Bad Transcript




◆ In the real world, for TBC $\tilde{P}$,

$$(T_i, X_i) = (T_j, X_j) \Rightarrow Y_i = Y_j$$
$$(T_i, Y_i) = (T_j, Y_j) \Rightarrow X_i = X_j$$

◆ There are conditions that can only hold in the ideal world:

$$(T_i, X_i) = (T_j, X_j) \wedge Y_i \neq Y_j$$
$$(T_i, Y_i) = (T_j, Y_j) \wedge X_i \neq X_j$$

- these conditions can hold at TBCs that are not used in the ideal world

◆ $\theta \in \mathrm{T}_{\mathrm{bad}}$ is a bad transcript if at least one of these conditions is satisfied

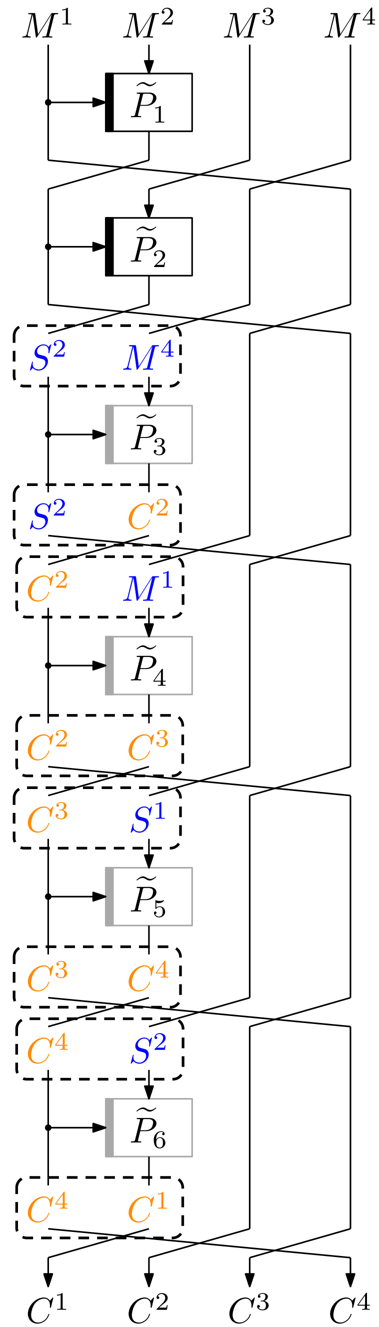
# Bad Transcript

- Example: (PRP proof, birthday-bound) Type-1 GFS with $d = 4, r = 2d - 2 = 6$

- $2n$-bit bad collisions can occur at $\tilde{P}_3, \dots, \tilde{P}_6$ that are not used in the ideal world
  - bad at $\tilde{P}_3$: $(S^2, M^4)$ and $(S^2, C^2)$
  - bad at $\tilde{P}_4$: $(C^2, M^1)$ and $(C^2, C^3)$
  - bad at $\tilde{P}_5$: $(C^3, S^1)$ and $(C^3, C^4)$
  - bad at $\tilde{P}_6$: $(C^4, S^2)$ and $(C^4, C^1)$

- We compute the probability of $\theta \in \mathrm{T}_{\mathrm{bad}}$ in the ideal world by taking summation of relevant bad probabilities.
  - For $r = 2d - 2$,

$$\Pr[\Theta_{\mathcal{J}} \in \mathrm{T}_{\mathrm{bad}}] \leq \frac{(d-1)q^2}{2^n} + \frac{0.5(d-1)q^2}{2^{2n}}$$



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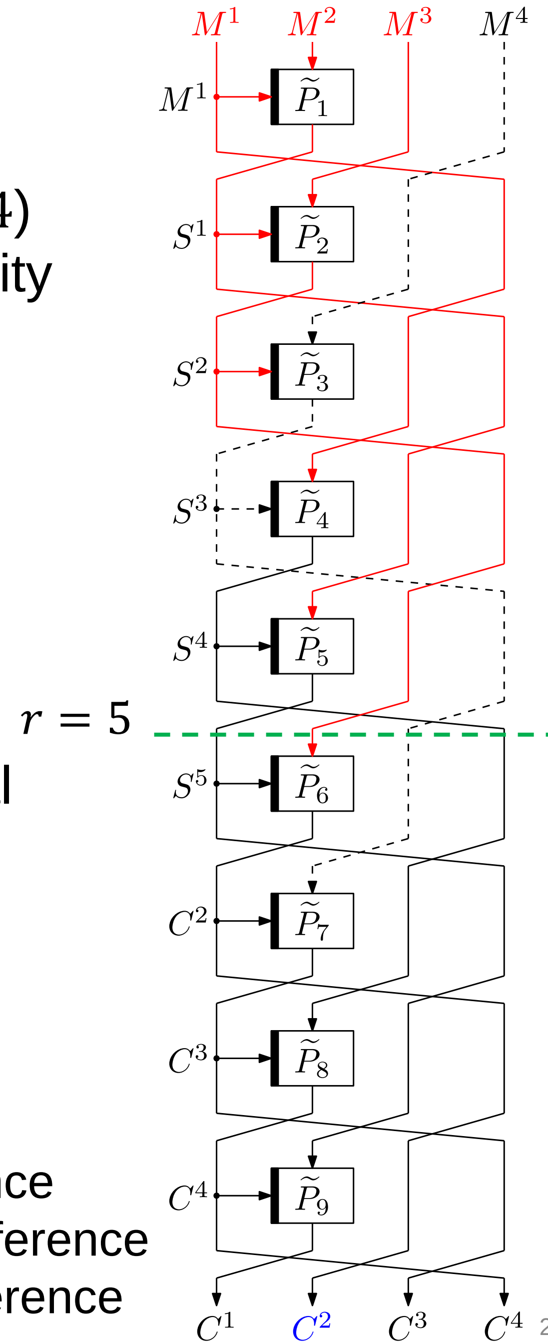
# Matching Attacks

- Example: CPA against Type-1 GFS ($d = 4$)
  - $r = 2d - 2 = 6$: birthday-bound security
  - $r = 3d - 2 = 10$: BBB security

- In the case $r < 6$:
  in the real world, a zero difference always exists in a ciphertext block
  $\Rightarrow$ distinguishable with 2 queries

  - implying that $r = 2d - 2$ is the optimal number of rounds for birthday-bound security



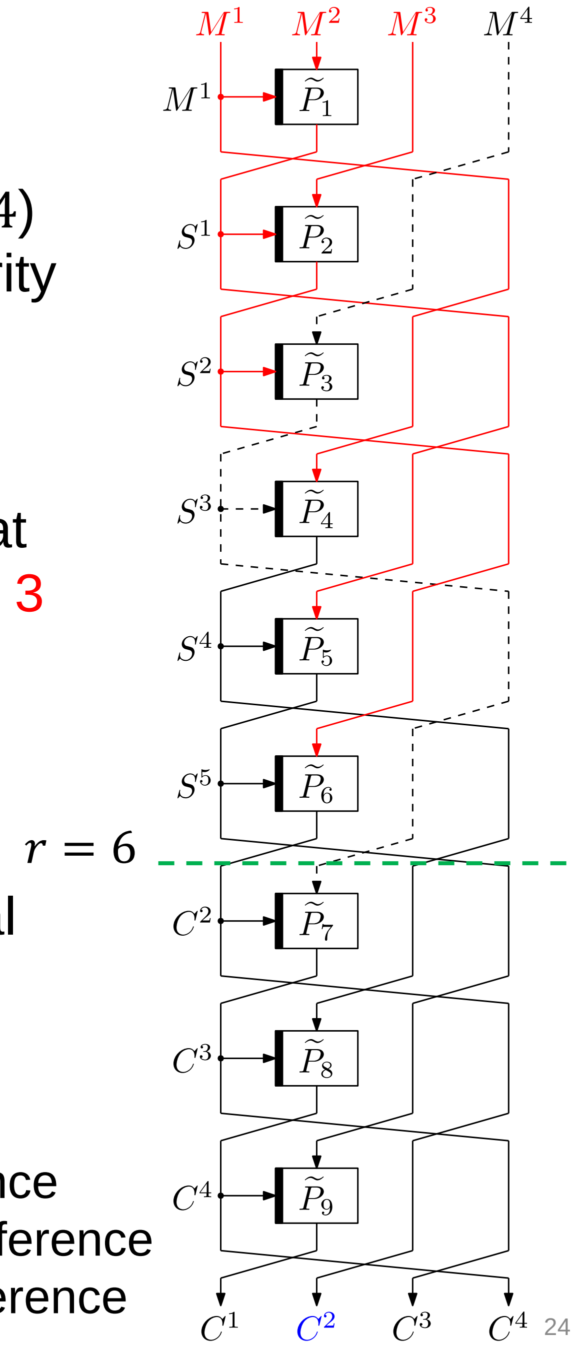

red: zero difference
dashed: non-zero difference
black: random difference

# Matching Attacks

- Example: CPA against Type-1 GFS ($d = 4$)
  - $r = 2d - 2 = 6$: birthday-bound security
  - $r = 3d - 2 = 10$: BBB security

- In the case $6 \leq r < 10$:
  in the real world, the collision probability at ciphertext block ($C^2$ in the figure) is about 3 times larger than in the ideal world (collision at $S^4$ or $S^5$ or $C^2$)
  ⇒ distinguishable with $2^{n/2}$ queries

  - implying that $r = 3d - 2$ is the optimal number of rounds for BBB security

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# Conclusions

- We formalized TBC-based type-1, type-2, and type-3 GFSs, and presented their provable security.
  - We identified the number of rounds to achieve birthday-bound security and BBB security.
- We also presented attacks to show the optimality of our results with respect to the number of rounds and attack complexity.
- Open questions
  - We do not know if an attack with $q = O(2^n)$ complexity exists when $r$ is larger than or equal to that for BBB security
  - stronger security bounds by increasing the number of rounds
  - indifferentiability of TBC-based GFSs

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# TBC calls for TBC-based GFSs

| | | The number of TBC calls | | # of parallel TBCs | |
|---|---|---|---|---|---|
| Const. | Model | for $r = r_{\mathrm{bb}}$ | for $r = r_{\mathrm{bbb}}$ | encryption | decryption |
| Type-1 | PRP | $2d - 2$ | $3d - 2$ | 1 | $d - 1$ |
| | SPRP | $d^2 - 2d + 2$ | $d^2 - d + 2$ | | |
| Type-2 | SPRP | $d^2/2$ | $d^2/2 + d$ | $d/2$ | $d/2$ |
| Type-3 | SPRP | $d^2 - d$ | $d^2 - 1$ | $d - 1$ | 1 |

- $r_{\mathrm{bb}}$ ($r_{\mathrm{bbb}}$): the number of rounds for birthday-bound security (BBB security)
- # of parallel TBCs: the number of TBCs that can be processed in parallel

- Example: when $r = r_{\mathrm{bb}}$ (SPRP),
  - if $d = 4$, # of TBC calls for Type-1 / 2 / 3 is 10 / 8 / 12
  - if $d = 8$, # of TBC calls for Type-1 / 2 / 3 is 50 / 32 / 56

    ⇒ Type-2 GFS has the smallest number of TBC calls