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Low-Latency Boolean Functions & Bijective S-boxes

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ESCADA

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- n- to m-bit vectorial Boolean functions

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- coordinate functions: f_i
- component functions: $F_\alpha := \langle F, \alpha \rangle \bigoplus_{i=0}^{m-1} \alpha_i f_i$ with $\alpha \in \mathbb{F}_2^m \setminus \{0\}$

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- each output value occurs uniformly, i.e., 2^{n-m} times

Introduction

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- n- to m-bit vectorial Boolean functions
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Balanced S-boxes

- each output value occurs uniformly, i.e., 2^{n-m} times
- equivalent to each component function is a balanced Boolean function

Cryptographic Properties

- uniformity: $$\mathrm{uni}(F) = \max_{\alpha \in \mathbb{F}_2^n \setminus \{0\},\, \beta \in \mathbb{F}_2^m} \#\{x \in \mathbb{F}_2^n \mid F(x) \oplus F(x \oplus \alpha) = \beta\}$$

Cryptographic Properties

- uniformity:
- linearity:

$$\mathrm{lin}(F) = \max_{\alpha \in \mathbb{F}_2^n,\, \beta \in \mathbb{F}_2^m \setminus \{0\}} \left| 2 \cdot \#\{x \in \mathbb{F}_2^n \mid \langle \alpha, x \rangle = F_\beta(x)\} - 2^n \right|$$

Cryptographic Properties

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- linearity:
- algebraic degree:

maximum number of input variables in each monomial of the ANF representation of each coordinate function

Properties

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Implementation Properties

- area
- gate count
- latency
- gate depth
- power
- ...

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Implementation Complexities

- gate count comp.
- gate depth comp.
- multiplicative comp.
- ...

Equivalency

$$F = P_{out} \circ G \circ P_{in}$$

- Bit-Permutation: with P_{in} and P_{out} being bijective bit-permutation functions

$$\# \text{ Eq. Fun. } \leq n! \cdot m!$$

Equivalency

- Bit-Permutation:

$$F = P_{out} \circ G \circ P_{in}(\cdot + \alpha) + \beta$$

with P_{in} and P_{out} being bijective bit-permutation functions

- Ext. Bit-Perm.:

$$\#\ \text{Eq. Fun.} \leq n! \cdot m! \cdot 2^{n+m}$$

Equivalency

- Bit-Permutation:
- Ext. Bit-Perm.:
- Linear:

$$F = L_{out} \circ G \circ L_{in}$$

with L_{in} and L_{out} being bijective linear functions

$\#$ Eq. Fun. $\leq \prod_{i=0}^{n-1}(2^n - 2^i) \cdot \prod_{i=0}^{m-1}(2^m - 2^i)$

Equivalency

- Bit-Permutation:
- Ext. Bit-Perm.:
- Linear:
- Affine:

$$F = A_{out} \circ G \circ A_{in}$$

with A_{in} and A_{out} being bijective affine functions

$$\#\ \text{Eq. Fun.} \leq \prod_{i=0}^{n-1}(2^n - 2^i) \cdot \prod_{i=0}^{m-1}(2^m - 2^i) \cdot 2^{n+m}$$

Equivalency

- Bit-Permutation:
- Ext. Bit-Perm.:
- Linear:
- Affine:
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$$F = A_{out} \circ G \circ A_{in} + L$$

with A_{in} and A_{out} being bijective affine functions and L being a linear function

$$\#\text{ Eq. Fun.} \leq \prod_{i=0}^{n-1}(2^n - 2^i) \cdot \prod_{i=0}^{m-1}(2^m - 2^i) \cdot 2^{nm+n+m}$$

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Extended Affine Equivalent Examples

- linearity / uniformity
- algebraic degree (of non-linear functions)
- multiplicative count / depth complexities

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- Linear:
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Bit-Permutation Equivalent Examples

- circuit implementation costs: area / latency / power
- gate depth / count complexities

- Bit-Permutation:
- Ext. Bit-Perm.:
- Linear:
- Affine:
- Ext. Affine:

$$F = A_{out} \circ G \circ A_{in} + L$$

with A_{in} and A_{out} being bijective affine functions and L being a linear function

$$\# \text{ Eq. Fun.} \leq \prod_{i=0}^{n-1}(2^n - 2^i) \cdot \prod_{i=0}^{m-1}(2^m - 2^i) \cdot 2^{nm+n+m}$$

Extended Bit-Permutation Semi-Equivalent Examples

- By accepting small tolerances,
- and due to combing small circuits/functions to build larger circuits/functions,

bit-perm. equivalent properties are extended bit-perm. semi-equivalent properties.

Latency

the time required to compute *all* the outputs of a circuit

Latency Complexity

Latency

the time required to compute *all* the outputs of a circuit

- circuit-specific property

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Gate Depth Complexity

the minimum possible value for the longest path (concerning the number of gates used in the path) from any input to any output for implementing the function

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in the basis of all gates with fan-in number 1 or 2

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Gate	NAND2	NOR2	AND2	OR2	XOR2	XNOR2
15 nm	2.031	2.554	3.580	3.644	5.268	6.788
45 nm	27.886	40.650	40.171	56.414	73.019	57.604

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Gate Depth Complexity

the minimum possible value for the longest path (concerning the number of gates used in the path) from any input to any output for implementing the function

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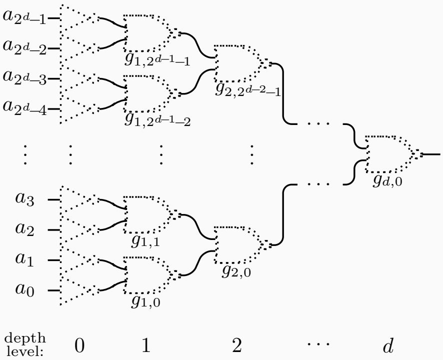
Latency Complexity

the gate depth complexity in the basis of $\{\mathtt{NAND2}, \mathtt{NOR2}, \mathtt{INV}\}$ without counting $\mathtt{INV}$s

Proposition 1

Any Boolean function $f(x_0, \ldots, x_{n-1})$ with latency complexity d can be implemented by a circuit of the following structure:



Building Representative Functions of $\mathcal{F}_{n,d}$

Data: $\mathcal{F}_{n',d'}$ for all $n' \leq n$ and $d' < d$ // the sets of all full-dependent n'-bit Boolean functions with latency complexity d'

Result: $\mathcal{F}_{n,d}$

$\mathcal{F} \leftarrow \emptyset$ and $\mathcal{F}_{n,d} \leftarrow \emptyset$

for $n_0 \leftarrow 1$ **to** n **do**

  for $n_1 \leftarrow \max(1, n - n_0)$ **to** n_0 **do**

    foreach $d_0, d_1 \in \mathbb{Z}_d$ **do**

      if $\mathcal{F}_{n_0,d_0} \neq \emptyset$ **and** $\mathcal{F}_{n_1,d_1} \neq \emptyset$ **and** $(d_0 = d - 1$ **or** $d_1 = d - 1)$ **then**

        foreach $\pi \in \mathbb{Z}_n^{n_1}$ **if** π *follows the restrictions* **do** // for $i < j$, $\pi[i] \neq \pi[j]$, // and if $n_0 \leq \pi_1[i]$ and $n_0 \leq \pi_1[j]$, then $\pi_1[i] < \pi_1[j]$.

          Compute the corresponding bit-permutation function P.

          foreach $\alpha \in \mathbb{F}_2^{n_1}$ **if** α *follows the restrictions* **do** // for each i such that $n_0 \leq \pi_1[i]$, $\alpha_1[i]$ must be 0.

            foreach $f_0^* \in \mathcal{F}_{n_0,d_0}$ **and** $f_1^* \in \mathcal{F}_{n_1,d_1}$ **do**

              $\mathcal{F} \leftarrow \mathcal{F} \cup \{f_0^* \barwedge f_1^*(P(\cdot) \oplus \alpha)\}$

foreach $f \in \mathcal{F}$ **do**

  $\mathcal{F}_{n,d} \leftarrow \mathcal{F}_{n,d} \cup \{\textsc{ComputeRepresentative}(f)\}$

for $n' \leftarrow 1$ **to** n **do**

  for $d' \leftarrow 0$ **to** d **do**

    if $(n', d') \neq (n, d)$ **then**

      $\mathcal{F}_{n,d} \leftarrow \mathcal{F}_{n,d} - \mathcal{F}_{n',d'}$

Number of Representative Functions in $\mathcal{F}_{n,d}$

d/n	2	3	4	5	6	7	8
1	1						
2	1	3	3				
3		5	54	159	170	64	20
4		2	149	109 674	20 658 457	227 737 882	?
5			2	506 005	?	?	?
6				66	?	?	?

Balanced

d/n	2	3	4	5	6	7	8
2	1	1					
3		2	6	8	3		
4		1	45	12 128	931 780	4 436 770	4 489 235
5			1	74 389	?	?	?
6				30	?	?	?

Finding All Possible Lowest-Depth Implementations of a Function

Data: $\mathcal{F}^*_{d'}$ for all d' `// all sets of` n`-bit representative Boolean functions (not necessarily full-dependent) with latency complexity` d'

Result: $\mathcal{I}$ `// all possible implementations of the given Boolean function`

Function FindAllCircuits(f):
- **if** $f = x_i$ *or* $f = \neg x_i$ **then** `// the case for latency complexity of` $d' = 0$
  - **return** the corresponding x_i or $\neg x_i$
- $\mathcal{I} \leftarrow \emptyset$, $\mathcal{A}_{\overline{\wedge}} \leftarrow \emptyset$ and $\mathcal{A}_{\overline{\vee}} \leftarrow \emptyset$
- **for** $d' \leftarrow 0$ **to** $d - 1$ **do**
  - **foreach** $g \in \{g \,|\, g \text{ is equivalent to } g^* \in \mathcal{F}^*_{d'}\}$ **do**
    - **if** $\neg f \wedge g = \neg f$ **then**
      - $\mathcal{A}_{\overline{\wedge}} \leftarrow \mathcal{A}_{\overline{\wedge}} \cup \{g\}$
    - **if** $\neg f \vee g = \neg f$ **then**
      - $\mathcal{A}_{\overline{\vee}} \leftarrow \mathcal{A}_{\overline{\vee}} \cup \{g\}$
  - **foreach** $g, h \in \mathcal{A}_{\overline{\wedge}}$ **do**
    - **if** $g \wedge h = \neg f$ **then**
      - $\mathcal{I} \leftarrow \mathcal{I} \cup \{(\text{FindAllCircuits}(g), \text{FindAllCircuits}(h), \texttt{NAND})\}$
  - **foreach** $g, h \in \mathcal{A}_{\overline{\vee}}$ **do**
    - **if** $g \vee h = \neg f$ **then**
      - $\mathcal{I} \leftarrow \mathcal{I} \cup \{(\text{FindAllCircuits}(g), \text{FindAllCircuits}(h), \texttt{NOR})\}$
  - **if** $\mathcal{I} \neq \emptyset$ **then** `// this means that the latency complexity of` f `is` $d' + 1$
    - **return** $\mathcal{I}$

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  χ_5 has $d = 3$ with $\ell = 16$ and $u = 8$,

  while APN S-boxes such as `Fides-5` have $d = 5$.

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  $\texttt{Speedy}$ S-box has $d = 4$ with $\ell = 24$ and $u = 8$.

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  `Speedy` S-box has $d = 4$ with $\ell = 24$ and $u = 8$.
- 7- and 8-bit:

  except χ_7 and χ_8 there is no S-box with $d \leq 5$.

Building S-boxes with Low-Latency Complexity

Criteria

1. latency complexity
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Method of Building S-boxes:

- Stepping over the coordinates of S-box
- $S = (f_0, f_1, \ldots, f_{n-1})$
- $\mathcal{F}$: the set of all Boolean functions satisfying the criteria 1, 2 (and 4)
- $\mathcal{R}_1$: the set of all representatives from $\mathcal{F}$

Building S-boxes with Steps on Coordinates [Can07]

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   - 3a. For each $F_0 \in \mathcal{R}_i$, check if F_1 is equivalent to F_0.

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   - 3a. For each $F_0 \in \mathcal{R}_i$, check if F_1 is equivalent to F_0.
     - If they are equivalent, set $\texttt{new} = 0$ and break the loop.

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   - 3b If $\texttt{new} = 1$, add F_1 to $\mathcal{R}_i$.
4. Increase i, if $i \leq m$. Set $\mathcal{S} = \emptyset$ and go to step 4.

Building S-boxes with Steps on Coordinates (Improved 1)

1. Set $i = 2$, and $\mathcal{R}_2 = \cdots = \mathcal{R}_m = \emptyset$
2. For each $F' \in \mathcal{R}_{i-1}$ and each $f \in \mathcal{F}$, compute $F = F' \parallel f$.

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   - If F fulfills the criteria, compute its representative and add it to $\mathcal{R}_i$.

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Improved Complexity

$$Time = \sum_{i=1}^{m-1} |\mathcal{F}| \cdot |\mathcal{R}_i| \cdot (t_{criteria\ check} + p_i \cdot t_{representative\ computation})$$

$$Memory = |\mathcal{F}| + \max_i |\mathcal{R}_i|$$

Building S-boxes with Steps on Coordinates (Improved 2)

1. Set $\mathcal{R}_2 = \cdots = \mathcal{R}_m = \emptyset$
2. For each $f_0 \in \mathcal{R}_1$ and each $f_1 \in \mathcal{F}$, compute $F_2 = f_0 \parallel f_1$.

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   - 2b. For each $f_2 \in \mathcal{F}$, compute $F_3 = F_2 \,||\, f_2$.

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     - If F_3 fulfills the criteria, compute its representative and add it to $\mathcal{R}_3$, otherwise, choose another f_2.
     - For each $f_3 \in \mathcal{F}$, compute $F_4 = F_3 \,\|\, f_3 \quad \ldots$

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     - If F_3 fulfills the criteria, compute its representative and add it to $\mathcal{R}_3$, otherwise, choose another f_2.
     - For each $f_3 \in \mathcal{F}$, compute $F_4 = F_3 \,\|\, f_3 \quad \ldots$

$$\begin{aligned} Time &= |\mathcal{R}_1| \cdot |\mathcal{F}| \cdot (t_0 + p_1 \cdot t_1 + p_1 \cdot |\mathcal{F}| \cdot (t_0 + p_2 \cdot t_1 + p_2 \cdot |\mathcal{F}| \cdot (\ldots))) \\ &= |\mathcal{R}_1| \cdot |\mathcal{F}| \cdot \big(t_0 \cdot (1 + p_1|\mathcal{F}| + p_1 p_2 |\mathcal{F}|^2 + \ldots) + t_1 p_1 \cdot (1 + p_2|\mathcal{F}| + \ldots)\big) \end{aligned}$$

$$Memory = |\mathcal{F}| + \sum_{i=1}^{m} |\mathcal{R}_i|$$

4. For each $f_0 \in \mathcal{R}_1$
   - 4a. Compute $\mathcal{F}_1^\dagger = \{f_1 \in \mathcal{F} \mid (f_0 \parallel f_1) \text{ fulfills the criteria}\}$

Building S-boxes with Steps on Coordinates (Improved 3)

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     2. Compute $\mathcal{F}_2^\dagger = \{f_2 \in \mathcal{F}_1^\dagger \mid (f_0 \parallel f_1 \parallel f_2) \text{ fulfills the criteria}\}$

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More Speed-Up Techniques

Ordering Coordinates

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- Uniformity: $\quad \mathrm{uni}(S_x) \leq 2^{m-x} \cdot \mathrm{uni}(S_x \| S_{m-x})$

Results

Latency Complexity $d = 2$

- 3-bit: 2 S-boxes with $\ell = 8$ and $u = 4$
- 4-bit: 1 S-box with $\ell = u = 16$
- 5-bit: 13 S-boxes with $\ell = u = 32$
- 6-bit: 19 S-boxes with $\ell = u = 64$
- 7-bit: 125 S-boxes with $\ell = u = 128$
- 8-bit: 181 S-boxes with $\ell = u = 256$

Latency Complexity $d = 3$

- 3-bit: all the S-boxes
- 4-bit: 281 S-boxes with $\ell = 8$ and $u = 4$
- 5-bit: 13 S-boxes with $\ell = 16$ and $u = 6$
- 6-bit: 49 quadratic S-boxes with $\ell = 32$ and $u = 16$
- 7-bit: 10 quadratic S-boxes with $\ell = 64$ and $u = 32$
- 8-bit: 84 quadratic S-boxes with $\ell = 128$ and $u = 64$

Latency Complexity $d = 4$

- 4-bit: all the Golden S-boxes: $\ell = 8$ and $u = 4$
- 5-bit: 2510 APN S-boxes: $\ell = 8$ and $u = 2$
- 6-bit: 908 quadratic S-boxes with $\ell = 16$ and $u = 4$
  together with one cubic S-box with $\ell = 16$ and $u = 4$ used in BipBip TBC
- 7-bit: 134 quadratic S-boxes with $\ell = 32$ and $u = 16$

Summary

- an algorithm to find all the Boolean functions with low-latency complexity
- an algorithm to find all the possible circuit corresponding to the latency complexity
- an improved algorithm for building S-boxes up-to the given criteria
- several low-latency S-boxes with good cryptographic properties
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Thank you for your attention!