New Bleichenbacher Records: Fault Attacks on qDSA Signatures

CHES 2018

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Outline

Introduction

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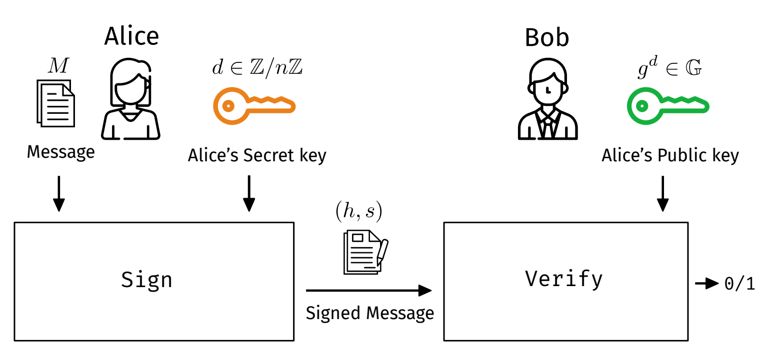
- Most notable variant: (EC)DSA
- Secure in ROM if the discrete logarithm problem (DLP) is hard
- Relies on an ephemeral random value known as nonce

Nonce in Schnorr Signatures



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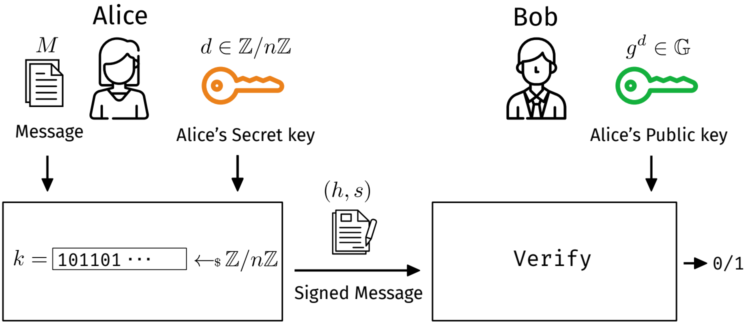

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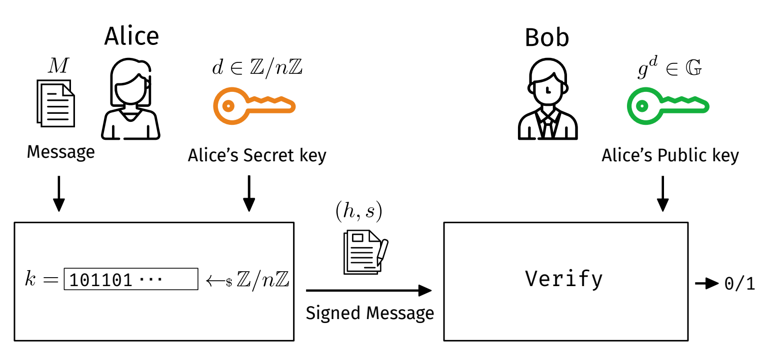

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$$k \equiv \underbrace{s}_{\text{public}} + \underbrace{h}_{\text{public}} d \mod n.$$

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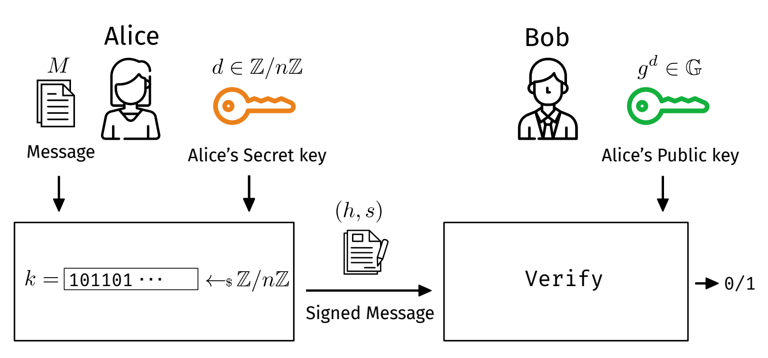

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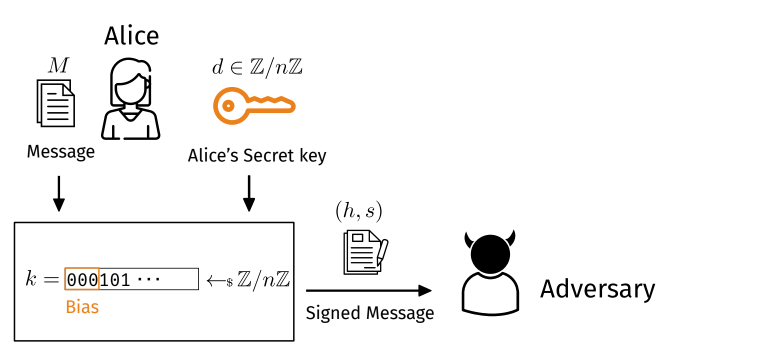
- k should NOT be reused/exposed

Risk of biased/leaky nonce



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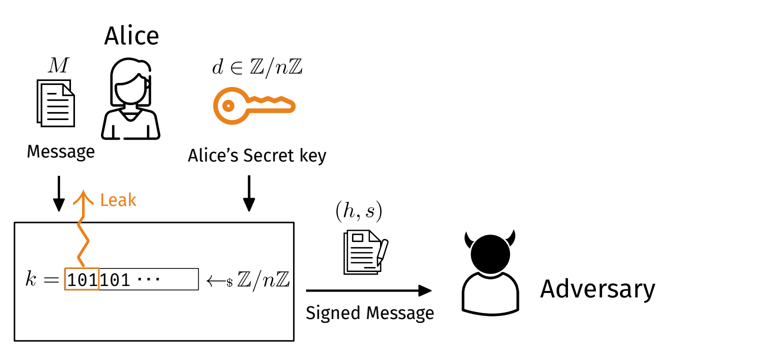

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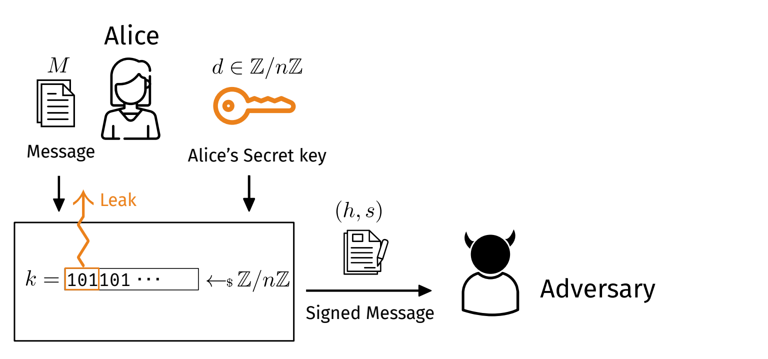

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⇝ Adversary could bypass the (EC)DLP and steal the secret d by solving the hidden number problem (HNP)!

Nonce: very sensitive!

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2. New fault attacks against recent, high-speed Schnorr-like signature scheme, qDSA, to obtain a few bits of nonces
3. Implemented a full secret key recovery attack against Schnorr-like signatures
    - Over 252-bit group
    - Only 2 or 3-bit nonce leaks

We set new records!

	# Leaked bits of Nonces				
	1	2	3	4	5
384-bit	–	–	–	–	[DMHMP14]
252-bit	–	–	–	–	–
160-bit	[AFG+14]	[LN13]	[NS02]	–	–

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- Orange: Bleichenbacher's attack
- Others: Lattice attack

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- Most important & costly phase is so-called range reduction of integers h
  - Necessary to detect the bias peak correctly and efficiently

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Looks like knapsack?

Difference: find many linear combinations instead of a single exact knapsack solution

Our Approach: Schroeppel-Shamir Algorithm

- Previous approaches are not optimal if the nonce bias is small:
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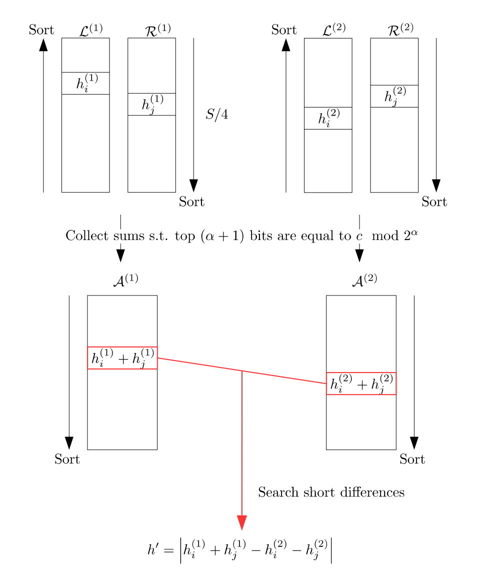
- Mentioned by Bleichenbacher, but never examined in the literature
- Advantages:
  - ☺ Highly space-efficient
  - ☺ Highly parallelizable with Howgrave-Graham–Joux's variant (EUROCRYPT'10, [HGJ10])

How HGJ–SS Helps

1. Split the inputs into four lists; sort.



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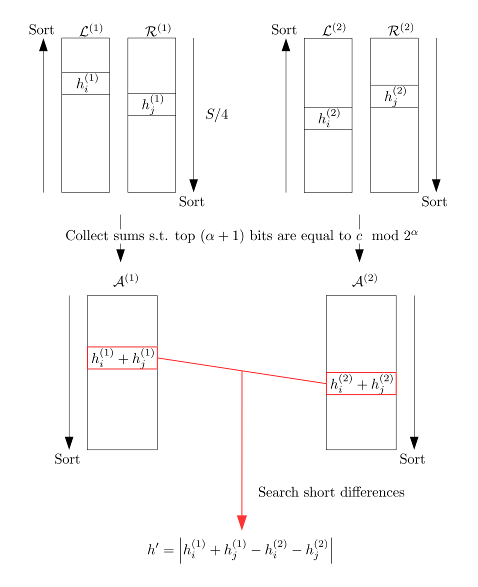

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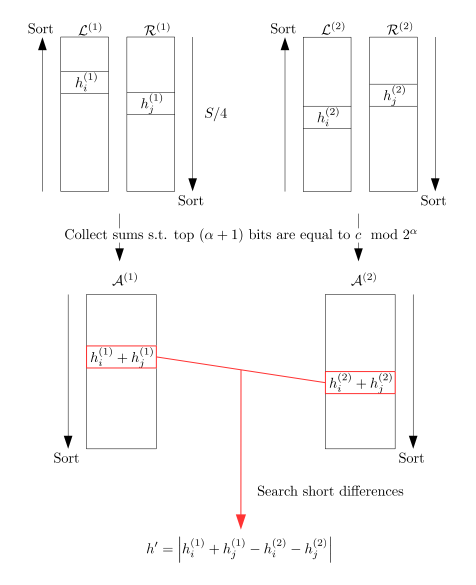
   → Get small LC's of 4 per round!



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HGJ–SS (1 round)	$\widetilde{O}(S^{4/3})$	$O(S^{2/3})$
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- HGJ–SS still terminates within a reasonable time frame due to parallelization

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  - By injecting a fault to the base point, we perturb it to non-prime/low-order points on Curve25519.

$$\pm\widetilde{R} \leftarrow \texttt{Ladder}(k, \pm\widetilde{P} = (\widetilde{X} : Z)) = \pm[k]\widetilde{P}$$

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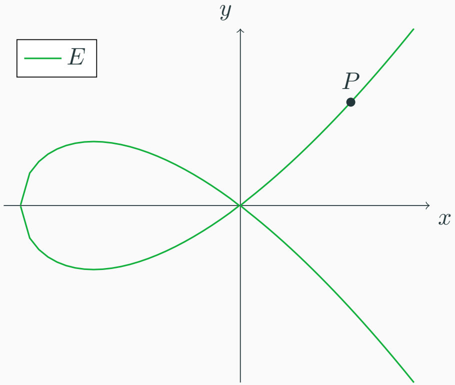

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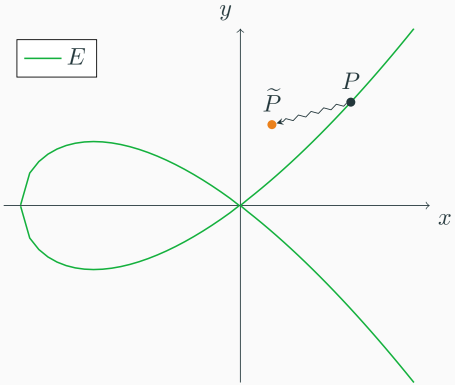

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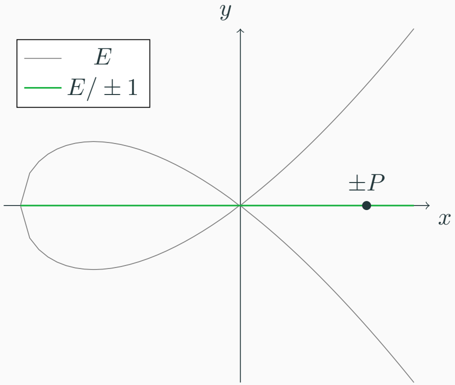

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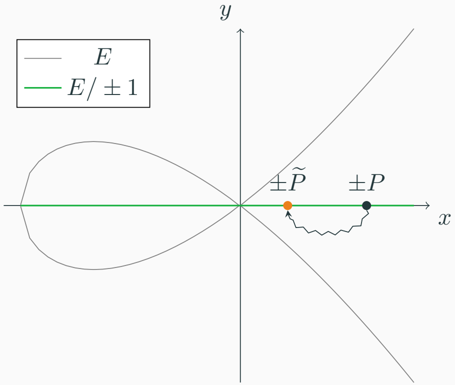

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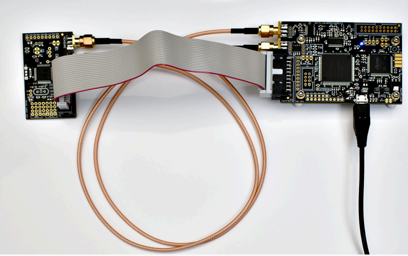
Fault Attacks on Curve25519 Base Point

1. Random semi-permanent fault against (program) memory
   ⇝ Can obtain 3-LSBs of nonce
2. Instruction skipping fault against base point initialization
   ⇝ Can obtain 2-LSBs of nonce
   - Verified using ChipWhisperer-Lite against AVR XMEGA



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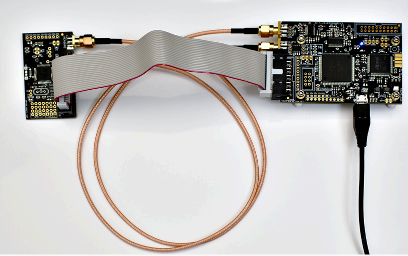

Countermeasure: multiply nonces by LCM of the the curve cofactor and the twist cofactor (i.e. "cofactor-killing")

$$\texttt{Ladder} : (8k, \pm\widetilde{P} = (\widetilde{X} : Z)) \mapsto \pm[8k]\widetilde{P}$$

Record-breaking Implementation of Nonce Attack

Result: Attack on 2-bit Leak

Wall clock time	CPU-time	Memory	# Sig	# MSB
16.7 days	11.7 years	15GB	2^{26}	26-bit

- Input: simulated 2^{45} faulty qDSA signatures, out of which 2^{26} instances (with $h <^{252-19}$) were fed into Bleichenbacher's attack

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- Estimation shows S&D would require at least 2^{35} inputs ≈ 2TB RAM!

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- Attacking with S&D is possible and faster , but requires much more signatures and RAM

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- First large-scale parallelization of Bleichenbacher
- Set new records!

Thank you!

Dank je!

GitHub: `https://github.com/security-kouza/new-bleichenbacher-records`

By Akira Takahashi, Mehdi Tibouchi, and Masayuki Abe

References I

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