

# Elastic MSM: A Fast, Elastic and Modular Preprocessing Technique for Multi-Scalar Multiplication Algorithm on GPUs

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**Abstract.** Zero-knowledge proof (ZKP) is a cryptographic primitive that enables a prover to convince a verifier that a statement is true, without revealing any other information beyond the correctness of the statement itself. Due to its powerful capabilities, its most practical type, called zero-knowledge Succinct Non-interactive ARGument of Knowledge (zkSNARK), has been widely deployed in various privacy-preserving applications such as cryptocurrencies and verifiable computation. Although state-of-the-art zkSNARKs are highly efficient for the verifier, the computational overhead for the prover is still orders of magnitude too high to warrant use in many applications. This overhead arises from several time-consuming operations, including large-scale matrix-vector multiplication (MUL), number-theoretic transform (NTT), and especially the multi-scalar multiplication (MSM) which constitutes the largest proportion. Therefore, further efficiency improvements are needed.

In this paper, we focus on comprehensive optimization of running time and storage space required by the MSM algorithm on GPUs. Specifically, we propose a novel, modular and adaptive parameter configuration technique—*elastic MSM* to enable us to adjust the scale of MSM according to our own wishes by performing a corresponding amount of preprocessing. This technique enables us to fully unleash the potential of various efficient parallel MSM algorithms. We have implemented and tested *elastic MSM* over three prevailing parallel Pippenger algorithms on GPUs. Across various preprocessing space limitations (across various MSM scales), our constructions achieve up to about 1.90×, 1.08× and 1.36× (2.58×, 1.39× and 1.91×) speedup versus three state-of-the-art parallel Pippenger algorithms on GPUs, respectively.

From another perspective, *elastic MSM* could also be regarded as a preprocessing technique over the well-known Pippenger algorithm, which is modular and could be used to accelerate almost all the most advanced parallel Pippenger algorithms on GPUs. Meanwhile, *elastic MSM* provides an adaptive trade-off between the running time and the extra storage space needed by parallel Pippenger algorithms on GPUs. This is the first preprocessing technique to retain the improved MSM computation brought by preprocessing under varying storage space limitations. Specifically, across various preprocessing space limitations (across various MSM scales), our constructions achieve up to about 192× and 223× (159× and 174×) speedup versus two state-of-the-art preprocessing parallel Pippenger algorithms on GPUs, respectively.

**Keywords:** Zero-Knowledge Proof · Multi-Scalar Multiplication (MSM) · Parallel Algorithm · Graphics Processing Unit (GPU)

# 1 Introduction

The proposal of a zero-knowledge argument system [GMR89], especially the *Non-Interactive Zero-Knowledge argument system* (NIZK) [BFM88], has a significant impact on both cryptography theory research and the application of cryptography. In the last decade, remarkable progress has been made in research on *zero-knowledge Succinct Non-interactive ARgument of Knowledge* (zkSNARK). After various studies [Kil92, Mic00, Gro10, Lip12, GGPR13], Groth constructed a pairing-based zkSNARK [Gro16] with only hundreds of bytes proof and is very fast to be verified within several milliseconds. Then, zkSNARKs are widely considered to be the most practical zero-knowledge proofs, and they have been widely applied to privacy-preserving applications, such as verifiable database outsourcing [ZGK<sup>+</sup>17], verifiable machine learning [ZFS20], privacy-preserving cryptocurrencies [BSCG<sup>+</sup>14, BG17, BMRS20], electronic voting [ZC16], online auction [GY19], and anonymous credentials [DLFKP16].

Although proofs of the most state-of-the-art zkSNARKs such as [Gro16, BBB<sup>+</sup>18, GWC19, CHM<sup>+</sup>20] are succinct and fast to verify, their generation remains a bottleneck in large-scale zkSNARK adoption. For example, to generate a proof for a program (which could be translated into a usually several times larger constraint system), the prover in these zkSNARKs need to perform various time-consuming operations, such as large-scale matrix-vector multiplication (MUL), number-theoretic transform (NTT), and the multi-scalar multiplication (MSM) on elliptic curves. And the number of operations required is always super-linear comparing to the number of constraints. As a consequence, it takes much longer to generate the zkSNARK proof of a program, and could be up to a few minutes just for a single payment transaction [BSCG<sup>+</sup>14]. Notably, among these expensive operations, [Xav22, LWY<sup>+</sup>23] have mentioned that for some of the most popular zkSNARK protocols, MSM is the most time-consuming operation, taking more than 70 percent of the total runtime.

In fact, there are already some efficient MSM algorithms available now, such as the Pippenger algorithm [Pip76], the Straus method [Str64], the Chang-Lou algorithm [CL03], and the Bos-Coster algorithm [dR95]. Especially, the Pippenger algorithm and its variants perform best when the scale of MSM is very large, as shown in [BDLO12]. These algorithms are still flourishing to this day. Recently, Botrel and Housni [BH23] optimized the arithmetic of finite fields by improving on the well-known Coarsely Integrated Operand Scanning (CIOS) modular multiplication and proposed a new coordinate system for twisted Edwards curves tailored for the Pippenger MSM algorithm. Luo, Fu and Gong [LFG23] proposed a bucket set construction that can be utilized in the context of Pippenger’s bucket method to speed up MSM over fixed points with the help of precomputation.

An important research line involves reducing the proof generation time by specifically designing MSM operations on certain hardware, including GPUs [Min19, Bel19, Spp22, Yrr22, Mat22, MXS<sup>+</sup>23, LWY<sup>+</sup>23, CPD<sup>+</sup>24], FPGAs [Xav22, ABC<sup>+</sup>22, Har22], ASICs [ZWZ<sup>+</sup>21], and CPU clusters [WZC<sup>+</sup>18]. Although DIZK [WZC<sup>+</sup>18] could accelerate the proof generation algorithm by distributing this algorithm to CPU clusters, it is still not suitable for widespread deployment due to much higher deployment overhead for CPU clusters than GPU cards, FPGA chips and ASIC chips. And due to extremely high development costs of ASIC, the ASIC design [ZWZ<sup>+</sup>21] is not yet widely adopted. Therefore, most recent research focused on improving the efficiency of MSM on GPUs or FPGAs.

## 1.1 Our Contributions

In this paper, our motivation is to design a new adaptive parameter configuration module, which is compatible with nearly all of the most efficient MSM algorithms currently available on GPUs, so as to further speed-up all these algorithms. Our contributions are as follows:

**Propose a new modular and adaptive parameter configuration technique.** With this series of research [Min19, Bel19, Spp22, Yrr22, Mat22, MXS<sup>+</sup>23, LWY<sup>+</sup>23], the fact that more and more potential of GPUs has been unleashed leads to better and better performance on the MSM parallel computations. Especially, the faster parallel MSM algorithm proposed in [LWY<sup>+</sup>23] has achieved nearly perfect linear speedup over the Pippenger algorithm [Pip76]. Thus, it seems challenging to further accelerate the parallel MSM algorithm solely through improvements to the MSM algorithm itself. As a result, we have shifted our focus towards the development of a new technique, termed *elastic MSM*. This technique enables us to change the scale of MSM (with sacrifice in storage space) adaptively. Notably, while different algorithms determine different computational complexities, our technique which is independent and compatible with algorithmic improvements endows us with the freedom to adjust parameters to maximize the power of a particular algorithm.

**Analyze the advantages of *elastic MSM* in theory.** Interestingly, we find that the *Computation Consolidation* technique proposed in [MXS<sup>+</sup>23] could be regarded as a special case of *elastic MSM*, that is *elastic MSM* with parameter fixed to a specific value. Therefore, our theory could provide a theoretical support for their technique from a higher dimension. With our plug-and-play technique *elastic MSM*, regardless of which Pippenger like algorithm is used, we can freely adjust the parameters of MSM instance according to the different parameter advantage intervals of each algorithm to fully unleash the potential of the algorithm. As an example, we combine our *elastic MSM* with some prevailing parallel MSM algorithms, to obtain some efficient preprocessing parallel Pippenger algorithms—*elastic Pippengers*, and theoretically analyze their advantages over parallel Pippenger algorithms without preprocessing, and the preprocessing parallel Pippenger algorithms obtained by combing these parallel MSM algorithms with the preprocessing technique proposed in [MXS<sup>+</sup>23]. The results show that *elastic Pippengers* have better time-space flexibility. Specifically, when operating within the same space limitations, *elastic MSM* exhibits faster preprocessing time and leads to more efficient MSM computation compared to the existing preprocessing method that also supports a flexible balance between time and space. Moreover, *elastic MSM* is the first preprocessing technique to retain the improved MSM computation brought by preprocessing under varying storage space limitations.

**Design some evaluation schemes for *elastic MSM*.** We plug both our preprocessing technique and the preprocessing technique proposed in GZKP [MXS<sup>+</sup>23] in two common parallel Pippenger algorithms on GPUs, to obtain four state-of-the-art preprocessing parallel Pippenger implementations on GPUs. The evaluation results demonstrate the effectiveness of our MSM algorithm. Specifically, with other practical parameters fixed:

1. Compared to the scheme with GZKP preprocessing:
  - Across preprocessed EC points from  $7 \cdot 2^{22}$  to  $2^{22}$ , our MSM algorithm achieves a speedup of approximately  $90\times$  and up to  $192\times$  in one implementation, and  $90\times$  and up to  $223\times$  in another implementation.
  - Across MSM scales from  $2^{16}$  to  $2^{22}$ , our MSM algorithm delivers a speedup of over  $16\times$  and up to  $159\times$  in one implementation, and  $32\times$  and up to  $174\times$  in another implementation.
2. Compared to the scheme without preprocessing:
  - Across preprocessed EC points from  $7 \cdot 2^{22}$  to  $2^{22}$ , our MSM algorithm achieves a speedup of over  $1.56\times$  and up to  $1.90\times$  in one implementation, and  $1.02\times$  and up to  $1.08\times$  in another implementation.
  - Across MSM scales from  $2^{16}$  to  $2^{22}$ , our MSM algorithm delivers a speedup of over  $1.70\times$  and up to  $2.58\times$  in one implementation, and  $1.05\times$  and up to  $1.39\times$  in another implementation.

The details of these evaluations are described in **Table 6**, **Table 7**, **Table 8** and **Table 9**.

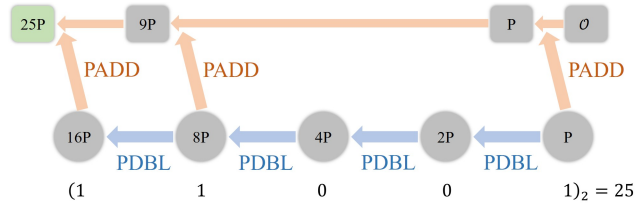
Moreover, we also plug our preprocessing technique in the most state-of-the-art parallel Pippenger algorithm [LWY<sup>+</sup>23], to obtain a faster preprocessing parallel Pippenger implementation on GPUs. With other practical parameters fixed, our MSM algorithm achieves a speedup of  $1.22\times$  to  $1.36\times$  when compared to the MSM algorithm described in [LWY<sup>+</sup>23] under varying storage space limitations. Additionally, across different MSM scales, our MSM algorithm delivers a speedup of  $1.33\times$  to  $1.91\times$ . Details of these evaluations are described in **Table 10**, **Table 11**.

## 2 Preliminaries

### 2.1 Elliptic curve

An elliptic curve (EC) is a smooth, projective, algebraic curve consisting of EC points. These points include the set that satisfies a specific mathematical equation, such as  $y^2 = x^3 + ax + b$ , and the point at infinity, denoted as  $\mathcal{O}$ , could be served as the identity element in the abelian group formed by all EC points.

These points support several common operations, including point addition (PADD), point doubling (PDBL) and point scalar multiplication (PMULT), where PADD is the fundamental operation. PDBL is a special case of PADD, the result of which is equal to performing a PADD operation on two identical points. As for PMULT of a scalar  $k$  and an EC point  $\mathbf{P}$ , it can be defined as  $k$  times self-PADD of  $\mathbf{P}$ , denoted by  $k\mathbf{P} = \mathbf{P} + \mathbf{P} + \dots + \mathbf{P}$ . In fact, the point  $k\mathbf{P}$  is always computed by using double-and-add method to execute a series of PDBLs and PADDs, we show an example of computing  $25\mathbf{P}$  in **Figure 1** to explain this method. We can first represent 25 in its binary form  $(11001)_2$  and initialize the result to be the point at infinity  $\mathcal{O}$ . Then at each bit position, we execute a PDBL to double the point. If the bit is 1, we add it to the result using a PADD.



**Figure 1:** An example of PMULT computation.  $\mathcal{O}$  is the point at infinity on elliptic curve.

### 2.2 Multi-scalar Multiplication

MSM performs the vector inner-product on the exponents of group elements. Specifically, the  $(n, \lambda)$ -MSM is defined as follows: given a vector of group elements  $(\mathbf{P}_1, \dots, \mathbf{P}_n)$ , we need to compute the formula  $Q = \sum_{i=1}^n k_i \mathbf{P}_i$ , using the minimal number of multiplications possible, where  $\{k_i\}_{i \in [1, n]}$  are all  $\lambda$ -bit scalars. If we compute MSM by performing PMULT for each  $k_i \mathbf{P}_i$  and then add all pairs  $k_i \mathbf{P}_i$  directly, it is clear that many expensive PMULT operations on an EC are needed. Based on our discussion in **Subsection 2.1**, if we simply employ the double-and-add method to compute MSM, we need to perform at most  $n\lambda + n - 1$  PADDs and  $n\lambda - n$  PDBLs. However in real-world applications, the security parameter  $\lambda$  commonly ranges from 254 to 768 and the scale of MSM  $n$  could be larger than a million and up to several billion. Even worse, the costs of EC point operations like PADD and PDBL themselves are much more expensive than regular scalar operations.

Altogether, the computational cost of simply using the double-and-add method for MSM calculation is not acceptable.

## 2.3 MSM in zkSNARK

zkSNARK is an application oriented variant of ZKP, which is one of the most important and widely used primitives in cryptography. A zkSNARK could be used by the prover to convince the verifier that, "given a public relation  $R$  and a public statement  $x$ , I know a secret witness  $w$  such that  $R(x, w) = 1$ ", without leaking any other information. While there are several different definitions about zkSNARK, we provide a brief introduction to the one known as *publicly-verifiable preprocessing zkSNARK* (see [BCI<sup>+</sup>13, GGPR13] for details). Informally, zkSNARK consists of three algorithms (Setup, Prove, Verify) (see **Figure 2**) such that

- $(pk_R, vk_R) \leftarrow \text{Setup}(R)$  : On input the relation  $R$ , this probabilistic algorithm output a proving key  $pk_R$  and the verification key  $vk_R$ . Both keys are public parameters, and they could be used to prove/verify any number of statements about relation  $R$ . That is, the Setup needs to be run only once for a specific relation  $R$ .
- $\pi \leftarrow \text{Prove}(R, pk_R, x, w)$  : For the public relation  $R$ , the probabilistic prover receives the proving key  $pk_R$ , a public input  $x$  for  $R$  and a secret input  $w$  for  $R$ , outputs a proof  $\pi$  to prove that "I know a secret  $w$  such that  $R(x, w) = 1$ ". Notably, the generation of  $\pi$  involves randomness that ensures its zero knowledge property.
- $0/1 \leftarrow \text{Verify}(R, vk_R, x, \pi)$  : For the public relation  $R$ , the deterministic verifier receives the verification key  $vk_R$ , a public input  $x$  for  $R$  and a proof  $\pi$ , outputs a decision bit 0/1 ("reject" or "accept").

zkSNARK should satisfy the following five properties: (1) *completeness*, i.e., honestly generated proof can be verified. (2) *knowledge soundness*, i.e., if the proof is verified by the verifier, then the prover must know the witness  $w$  such that  $R(x, w) = 1$ . (3) *zero knowledge*, i.e., there is no any other information could be derived from the proof other than that the statement  $x$  is true. (4) *succinctness*, i.e., small proof sizes and fast verification regardless of  $R$ 's complexity. (5) *non-interactive*, i.e., only a single message from the prover to the verifier.

It can be seen from the definition that the design intention of zkSNARK determines that itself has a small proof size and a fast verification. This allows zkSNARKs to serve as a foundation for highly efficient verifiable computation. However, the trade-off in the prover's complexity becomes a computational bottleneck. If we take apart the computation of the prover, just as shown in the data of [Xav22, LWY<sup>+</sup>23], the MSM operation accounts for the vast majority of the proportion (more than 70% for many state-of-the-art zkSNARKs), making it the main computational bottleneck of the zkSNARK prover's complexity.

## 2.4 The Pippenger Algorithm

Four decades ago, Pippenger provided an asymptotically optimal algorithm for the MSM in [Pip76]. To this day, the Pippenger algorithm and its variants are still the state-of-the-art and widely used algorithms. It is noticeable that our *elastic MSM* technique could be compatible with all the Pippenger-like parallel MSM algorithms. In this subsection, we review the Pippenger algorithm and analyze its computational costs (also see a simple example in **Figure 3**).

**Algorithm Description.** If we set the scale of MSM to be  $n$ , the bit length of scalar to be  $\lambda$  and the window size to be  $s$  (without losing generality, we assume that  $\lambda$  is divisible by  $s$  in our paper), then in order to compute the MSM  $\mathbf{Q} = \sum_{i=1}^n \mathbf{Q}_i = \sum_{i=1}^n k_i \mathbf{P}_i$ , the Pippenger algorithm consists of these three steps:

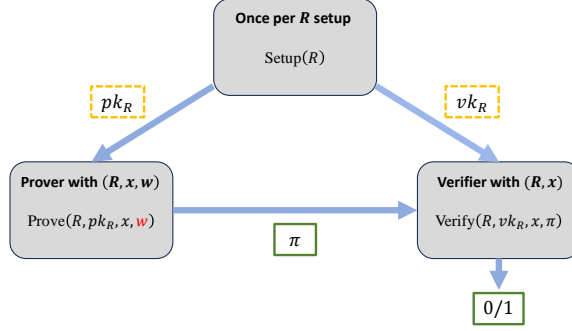


Figure 2: Components of a zkSNARK.

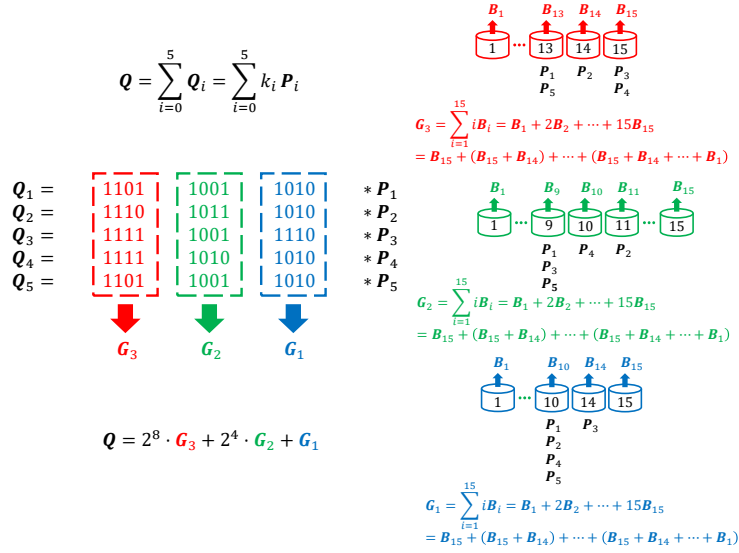


Figure 3: An example of Pippenger algorithm with 5 MSM scale, 12-bit scalars, and window size 4.

1. **Convert the task using *windowed MSM* technique.** We first divide the original task  $\mathbf{Q} = \sum_{i=1}^n k_i \mathbf{P}_i$  into multiple smaller subtasks according to the window, and we denote this step by *windowed MSM*. With the window size  $s$ , it is clear that we could divide each  $\lambda$ -bit scalars  $k_i$  into  $\frac{\lambda}{s}$  parts, and each part is an  $s$ -bit scalars  $m_{ij}$ , satisfying  $k_i = \sum_{j=1}^{\frac{\lambda}{s}} 2^{(j-1)s} m_{ij}$ . Then the smaller subtasks are defined as the computation  $\mathbf{G}_j = \sum_{i=1}^n m_{ij} \mathbf{P}_i$ , where  $j \in [1, \frac{\lambda}{s}]$ . Therefore, the relation between the original task and these subtasks can be expressed by **Formula (1)**

$$\begin{aligned}
 \mathbf{Q} &= \sum_{i=1}^n \mathbf{Q}_i = \sum_{i=1}^n k_i \mathbf{P}_i = \sum_{i=1}^n \sum_{j=1}^{\frac{\lambda}{s}} (2^{(j-1)s} m_{ij}) \mathbf{P}_i \\
 &= \sum_{j=1}^{\frac{\lambda}{s}} 2^{(j-1)s} \sum_{i=1}^n m_{ij} \mathbf{P}_i = \sum_{j=1}^{\frac{\lambda}{s}} 2^{(j-1)s} \mathbf{G}_j
 \end{aligned} \tag{1}$$

2. **Compute subtask results  $\mathbf{G}_j$ , for each  $j \in [1, \frac{\lambda}{s}]$ .** In each subtask (indexed by  $j$ ), we put EC points  $\mathbf{P}_i$  with the same scalar value  $m_{ij}$  into the specific bucket whose index is equal to  $m_{ij}$ . Notably, the points corresponding to zero scalars have no effect on the final result, so there are only  $2^s - 1$  buckets needed. Then to compute the subtask results  $\mathbf{G}_j$ , we only need to perform the following two steps:
  - (a) Sum all points in the bucket to obtain  $\mathbf{B}_l^{(j)}$ , for each  $l \in [1, 2^s - 1]$ . It is obvious that in each subtask, the sum of all points in the buckets requires at most  $n$  PADDs.
  - (b) Sum all bucket points weighted by their bucket indexes, namely  $\mathbf{G}_j = \sum_{l=1}^{2^s-1} l\mathbf{B}_l^{(j)}$ , to obtain  $\mathbf{G}_j$ . Obviously,  $\sum_{l=1}^{2^s-1} l\mathbf{B}_l^{(j)} = \sum_{i=1}^n m_{ij}\mathbf{P}_i = \mathbf{G}_j$ . And by using an efficient algorithm proposed in [BDLO12] (as shown in **Algorithm 1**), the computation of subtask result  $\mathbf{G}_j$  requires at most  $2^{s+1} - 2$  PADDs.
3. **Compute the MSM result with subtask results, namely  $\mathbf{Q} = \sum_{j=1}^{\frac{\lambda}{s}} 2^{(j-1)s}\mathbf{G}_j$ .** Note that:

$$\mathbf{Q} = \sum_{j=1}^{\frac{\lambda}{s}} 2^{(j-1)s}\mathbf{G}_j = 2^s \left( \cdots \left( 2^s \left( 2^s \mathbf{G}_{\frac{\lambda}{s}} + \mathbf{G}_{\frac{\lambda}{s}-1} \right) + \mathbf{G}_{\frac{\lambda}{s}-2} \right) \cdots \right) + \mathbf{G}_1 \quad (2)$$

So, computing the MSM result requires at most  $(\frac{\lambda}{s} - 1)s$  PDBLs and  $\frac{\lambda}{s}$  PADDs. Then this computation requires at most  $\lambda - s$  PDBLs and  $\frac{\lambda}{s}$  PADDs.

In summary, for each subtask, it requires at most  $n$  PADDs to put all points into the buckets and  $2^{s+1} - 2$  PADDs to get the subtask result using **Algorithm 1**. And to add the subtask results to the final result, a recursive method based on **Formula (2)** can be used, which requires around  $\lambda - s$  PDBLs and  $\frac{\lambda}{s}$  PADDs. Since there are  $\frac{\lambda}{s}$  subtasks, the total computational costs of the Pippenger algorithm are around  $\frac{\lambda}{s}(n + 2^{s+1})$  PADDs plus  $\lambda - s$  PDBLs. Note that the value of  $\frac{\lambda}{s}$  and  $s$  are usually small, so it is common for us to omit them in the addition or subtraction terms. However, we explicitly express  $s$  in the complexity analysis of PDBLs because it will affect our discussion in **Subsection 3.2**. Moreover, we skip the costs of scalar operations here because they are negligible compared to the costs of EC point operations.

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**Algorithm 1** BucketPointsReduction [BDLO12]
 

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**Require:** A point vector  $\vec{\mathbf{B}}_{2^s-1}^{(j)} = [\mathbf{B}_1^{(j)}, \mathbf{B}_2^{(j)}, \dots, \mathbf{B}_{2^s-1}^{(j)}]$

**Ensure:**  $\mathbf{G}_j = \sum_{l=1}^{2^s-1} l\mathbf{B}_l^{(j)}$

- 1:  $\mathbf{G}_{j,0} \leftarrow \mathcal{O}; \mathbf{M}_0 \leftarrow \mathcal{O}$  //  $\mathcal{O}$  is the point at infinity on the elliptic curve.
  - 2: **for**  $l \leftarrow 1$  to  $2^s - 1$  **do** // Add  $l\mathbf{B}_l^{(j)}$  as
  - 3:    $\mathbf{M}_l \leftarrow \mathbf{M}_{l-1} + \mathbf{B}_{2^s-l}^{(j)}$  //  $\mathbf{M}_l = \mathbf{B}_{2^s-1}^{(j)} + \mathbf{B}_{2^s-2}^{(j)} + \cdots + \mathbf{B}_{2^s-l}^{(j)}$
  - 4:    $\mathbf{G}_{j,l} \leftarrow \mathbf{G}_{j,l-1} + \mathbf{M}_l$  //  $\mathbf{G}_{j,l} = \mathbf{M}_1 + \mathbf{M}_2 + \cdots + \mathbf{M}_l$
  - 5: **end for**
  - 6:  $\mathbf{G}_j \leftarrow \mathbf{G}_{j,2^s-1}$
  - 7: **return**  $\mathbf{G}_j$
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## 2.5 Graphics Processing Units

Graphics Processing Units (GPUs) are platforms composed of hundreds or even thousands of cores that can handle thousands of threads simultaneously. This makes them particularly



well-suited for tasks that can be broken down into many smaller tasks that can be executed in parallel, such as matrix operations, convolutional neural networks, and physical simulations. A typical GPU consists of multiple Streaming Multiprocessors (SMs) and a global memory. Each SM includes multiple Scalar Processors (SPs), a shared memory, and several on-chip registers. These registers and various kinds of memory constitute the multiple memory hierarchy architecture of GPUs. The on-chip registers are the fastest memory component but have minimal storage capacity, while the global memory provides the largest storage capacity but is the slowest. The performance of the shared memory is between the on-chip registers and the global memory. GPUs has a special execution model—Single Instruction, Multiple Threads (SIMT) execution model that executes batches of threads in lockstep. In the SIMT model, threads executing the same instruction are grouped into a fixed-sized batch, called a wavefront (AMD) or a warp (NVIDIA). The threads of a batch always execute the same instruction in lockstep on a single instruction, multiple data (SIMD) unit, i.e., in parallel on different operands.

### 3 A New Parameter Configuration Technique

Recently, there are many excellent works such as [Min19, Bel19, Spp22, Yrr22, Mat22, MXS<sup>+</sup>23, LWY<sup>+</sup>23] trying to optimize the parallel Pippenger algorithm itself. Especially, the faster parallel Pippenger-like algorithm proposed in [LWY<sup>+</sup>23] has achieved nearly perfect linear speedup over the Pippenger algorithm [Pip76], where perfect linear speedup means the parallel speedup ratio is equal to the number of execution threads. Although this road is thriving, we find that there is another parallel shortcut to efficiency that is often overlooked.

In our opinion, the optimization of parallel MSM algorithms could be divided into two independent and mutually beneficial directions. While one direction focuses on MSM algorithm optimization itself, which is currently a popular direction. The other direction concentrates on techniques to optimize and adjust the scale of MSM to better adapt to a certain MSM algorithm, which is exactly what we delve into in this paper. In fact, as shown in [BDLO12], the Pippenger algorithm performs best when the scale of MSM is very large. Also as noticed in [LWY<sup>+</sup>23], the advantage of Pippenger algorithm only comes when there are a great amount of EC points placed into the same buckets and processed as a whole, which means that the larger the scale of MSM is, the more benefits this advantage brings.

#### 3.1 The Elastic MSM Algorithm

Recall that the first step of the Pippenger algorithm is to divide the original task into multiple smaller subtasks, using *windowed MSM* technique. In this subsection, we present a novel, modular and adaptive parameter configuration technique—*elastic MSM*, which, like *windowed MSM*, could also be regarded as a task partitioning scheme.

From **Formula (1)**, it is obvious that we could deduce the equation  $\mathbf{Q} = \sum_{i=1}^n \mathbf{Q}_i = \sum_{i=1}^n \sum_{j=1}^{\frac{\lambda}{s}} (2^{(j-1)s} m_{ij}) \mathbf{P}_i$ , thus for each  $i \in [1, n]$  we have  $\mathbf{Q}_i = \sum_{j=1}^{\frac{\lambda}{s}} (2^{(j-1)s} m_{ij}) \mathbf{P}_i$ . Now we define the  $w, k$  such that  $\frac{\lambda}{s} = w \cdot k$ , this equation could be equivalently expressed in the form of a matrix multiplication:

$$\mathbf{Q}_i = \begin{pmatrix} 1 & 2^s & 2^{2s} & \dots & 2^{(wk-1)s} \end{pmatrix} \cdot \begin{pmatrix} m_{i1} \mathbf{P}_i \\ m_{i2} \mathbf{P}_i \\ m_{i3} \mathbf{P}_i \\ \vdots \\ m_{i(wk)} \mathbf{P}_i \end{pmatrix} \quad (3)$$



With the  $\frac{\lambda}{s} = w \cdot k$ , our ideas come from an observation that for each  $i \in [1, n]$ , if we define  $M_{i(l,t)} := m_{i((l-1)k+t)}$ , where  $l \in [1, w]$  and  $t \in [1, k]$ , then the following equation holds:

$$\mathbf{Q}_i = \sum_{j=1}^{wk} \left( 2^{(j-1)s} m_{ij} \right) \mathbf{P}_i = \sum_{l=1}^w \sum_{t=1}^k 2^{((l-1)k+(t-1))s} M_{i(l,t)} \mathbf{P}_i$$

This means that except the **Formula (3)**,  $\mathbf{Q}_i$  could also be equivalently expressed as another form of matrix multiplication:

$$\begin{pmatrix} 1 & 2^{ks} & \cdots & 2^{(w-1)ks} \end{pmatrix} \cdot \mathbf{P}_i \cdot \begin{pmatrix} M_{i(1,1)} & M_{i(1,2)} & \cdots & M_{i(1,k)} \\ M_{i(2,1)} & M_{i(2,2)} & \cdots & M_{i(2,k)} \\ \vdots & \vdots & & \vdots \\ M_{i(w,1)} & M_{i(w,2)} & \cdots & M_{i(w,k)} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2^s \\ \vdots \\ 2^{(k-1)s} \end{pmatrix} \quad (4)$$

If we define  $\mathbf{P}_{ij} := 2^{(j-1)ks} \mathbf{P}_i$ ,  $\mathbf{G}_{il} := \sum_{j=1}^w M_{i(j,l)} \mathbf{P}_{ij}$  and  $N_{ij} := \sum_{l=1}^k 2^{(l-1)s} M_{i(j,l)}$ , where  $i \in [1, n]$ ,  $j \in [1, w]$  and  $l \in [1, k]$ , then the **Formula (4)** can be deduced to the **Formula (5)**, which is actually a formula similar to the **Formula (1)**.

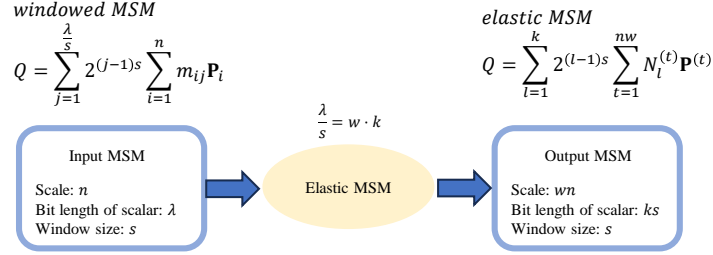
$$\begin{aligned} \mathbf{Q}_i &= \sum_{j=1}^w N_{ij} \mathbf{P}_{ij} = \sum_{j=1}^w \sum_{l=1}^k 2^{(l-1)s} M_{i(j,l)} \mathbf{P}_{ij} \\ &= \sum_{l=1}^k 2^{(l-1)s} \sum_{j=1}^w M_{i(j,l)} \mathbf{P}_{ij} \\ &= \sum_{l=1}^k 2^{(l-1)s} \mathbf{G}_{il} \end{aligned} \quad (5)$$

By summarizing the above derivation, we then obtain our core technique—*elastic MSM*. For  $\frac{\lambda}{s} = w \cdot k$ ,  $i \in [1, n]$ , and  $j \in [1, w]$  and  $l \in [1, k]$ , if we denote  $N_l^{((i-1)w+j)} := M_{i(j,l)}$ ,  $\mathbf{P}^{((i-1)w+j)} := \mathbf{P}_{ij}$ , and  $\mathbf{G}^{(l)} := \sum_{t=1}^{nw} N_l^{(t)} \mathbf{P}^{(t)}$ , then the whole computation of MSM can be represented by **Formula (6)**.

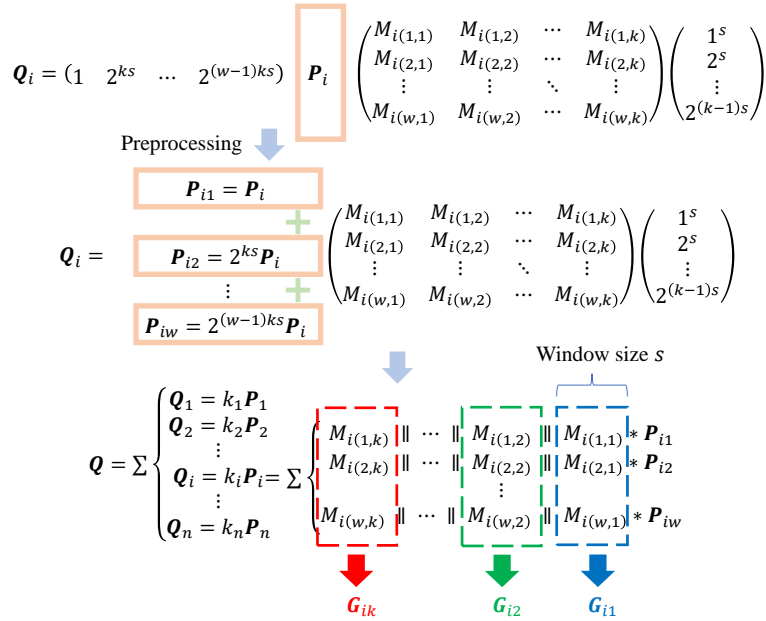
$$\begin{aligned} \mathbf{Q} &= \sum_{i=1}^n \mathbf{Q}_i = \sum_{i=1}^n \sum_{j=1}^w N_{ij} \mathbf{P}_{ij} = \sum_{i=1}^n \sum_{j=1}^w \sum_{l=1}^k 2^{(l-1)s} M_{i(j,l)} \mathbf{P}_{ij} \\ &= \sum_{t=1}^{nw} \sum_{l=1}^k 2^{(l-1)s} N_l^{(t)} \mathbf{P}^{(t)} = \sum_{l=1}^k 2^{(l-1)s} \sum_{t=1}^{nw} N_l^{(t)} \mathbf{P}^{(t)} \\ &= \sum_{l=1}^k 2^{(l-1)s} \mathbf{G}^{(l)} \end{aligned} \quad (6)$$

Note that this **Formula (6)** obtained from our *elastic MSM* is very similar to the **Formula (1)** obtained from the trivial *windowed MSM*. So we could also perform the Pippenger algorithm according to the **Formula (6)** rather than **Formula (1)**. Specifically, we divide the coefficients of each  $\mathbf{Q}_i$  into windows and express them as a  $w \times k$  matrix (each element in this matrix is still an  $s$ -bit window), instead of expressing them as a long vector just like *windowed MSM*. That is, with  $\frac{\lambda}{s} = w \cdot k$ , both our *elastic MSM* and *windowed MSM* choose the same window size  $s$ . However, we consider the scale of MSM as  $wn$  rather than  $n$  and the bit length of scalar as  $ks$  rather than  $\lambda$  (see **Figure 4** and **Figure 5** for easier understanding). Moreover, the computation in our *elastic MSM* is

based on the vectors of EC points  $\mathbf{P}_{ij} := 2^{(j-1)ks}\mathbf{P}_i$ , rather than the original vectors of EC points  $\mathbf{P}_i$ , for  $i \in [1, n]$  and  $j \in [1, w]$ . Fortunately, given points  $\{\mathbf{P}_i\}_{i \in [1, n]}$ , all the  $\{\mathbf{P}_{ij}\}_{i \in [1, n], j \in [1, w]}$  could be pre-computed. Thus, our *elastic MSM* could be regarded as a novel, modular and adaptive parameter configuration technique, which could be used in all the Pippenger-like algorithms to replace the *windowed MSM* technique. In this way, it could provide a flexible adjustment between the scale of MSM and bit length of scalar, but at the cost of preprocessing the original vectors of EC points.



**Figure 4:** An interpretation of instance conversion.



**Figure 5:** A demonstration of *elastic MSM*

**Preprocessing complexity.** Throughout this paper, we measure memory space overhead in theory by the number of extra points that need to be stored. Given the points  $\{\mathbf{P}_i\}_{i \in [1, n]}$ , for each  $i \in [1, n]$ , we need to additionally preprocess and store the points  $\{2^{jks}\mathbf{P}_i\}_{j \in [1, w-1]}$ , to obtain all the  $\{\mathbf{P}_{ij}\}_{i \in [1, n], j \in [1, w]}$ . Such processing requires  $(w-1)ks$  PDBLs and  $w-1$  extra memory space overhead for each  $i \in [1, n]$ , and  $n(w-1)ks$  PDBLs and  $n(w-1)$  extra memory space overhead in total for all  $i \in [1, n]$ . Therefore, from another perspective, our solution actually provides a trade-off between time and space saving. When our storage space is limited, we can flexibly adjust the storage space requirement from 0 point to all  $n(\frac{\lambda}{s}-1)$  points by adjusting the parameter  $w$  and corresponding  $k$ .

**Additional overhead.** Notably, the *elastic MSM* requires us to convert a one-dimensional vector into a two-dimensional matrix, which may introduce additional transformation overhead. This could also be achieved through an irregular access pattern, which involves accessing this one-dimensional vector with a stride of  $k$ . With these implementation methods, we observed that the additional overhead arises primarily from exceptionally fast operations (such as some bit operations and data access), as opposed to the computationally expensive elliptic curve operations. Specifically, when the MSM scale is set to  $2^{16}$  (or  $2^{22}$ ), the incremental overhead for this component amounts to about 0.001 seconds (or 0.009 seconds). Consequently, its contribution to the overall time complexity is minimal.

**About constant time.** Note that the execution times of both the Pippenger algorithm and the *elastic MSM* algorithm are not constant. In other words, they are heavily time-dependent. Specifically, the time of Pippenger is related to parameters  $\lambda, s$  and  $n$ . And the time of *elastic MSM* is related to parameters  $w, k, s$  and  $n$ , where  $\frac{\lambda}{s} = w \cdot k$ . Clearly, the time of *elastic MSM* could also be considered to be related to parameters  $\lambda$  and  $s$ , as both freely variable  $w$  and  $k$  are constrained by  $\lambda$  and  $s$ .

Now, we explain why these heavily time-dependent algorithms do not result in any leakage about the secret input. Recall that the  $(n, \lambda)$ -MSM is defined as follows: given a vector of group elements  $(\mathbf{P}_1, \dots, \mathbf{P}_n)$ , we need to compute the formula  $Q = \sum_{i=1}^n k_i \mathbf{P}_i$ , using the minimal number of multiplications possible, where  $\{k_i\}_{i \in [1, n]}$  are all  $\lambda$ -bit scalars. In the zkSNARK construction, we may designate some of the input/output wires as specifying a public input, and use the rest of the wires in the circuit to denote a secret input. This gives us a binary relation  $R$  consisting of public wires and secret wires that satisfy the arithmetic circuit. This set of  $\{k_i\}_{i \in [1, n]}$  could be regarded as a set of assignments to all the wires in the circuit corresponding to the relation  $R$ . Thus, w.l.o.g., we could consider the assignments on the public wires as  $\{k_i\}_{i \in [1, l]}$  and the assignments on the secret wires as  $\{k_i\}_{i \in [l+1, n]}$  for a certain  $l$ . The parameter  $\lambda$  is determined by a specific elliptic curve, which is public and independent of  $\{k_i\}_{i \in [l+1, n]}$ . The parameter  $s$  is a freely variable parameter, which is clearly public and independent of  $\{k_i\}_{i \in [l+1, n]}$ . In the zkSNARK construction, the parameter  $n$  is total number of wires in the circuit corresponding to the relation  $R$ . The relation  $R$  is public, and then the corresponding circuit is also public, so  $n$  is naturally public. Moreover,  $n$  is only related to the circuit (that is, the relation  $R$ ) itself and not to the specific assignments, so  $n$  is also independent of  $\{k_i\}_{i \in [l+1, n]}$ . That's also why  $(\mathbf{P}_1, \dots, \mathbf{P}_n)$  can be publicly given to us. Furthermore, although 0 scalars have a little impact on the time of summing all points in buckets, this problem can be solved by just performing addition operations on EC points at infinity.

Very recently, we note that the concurrent work [CPD<sup>+</sup>24] mentions that the bucket accumulation phase is not constant-time, meaning its runtime depends on the secret input. This conclusion seems to contradict our discussion, so we aim to provide clarifications here. Let's denote the total number of threads as  $t$ . Their scheme recommends a large value for  $s$ , specifically  $s = 16$ . However, the large  $s$  necessitates reducing the space overhead during MSM computation (otherwise,  $t \cdot (2^{16} - 1)$  bucket space would be required). This scheme adopts a method where multiple threads share bucket space, thereby necessitating the resolution of write conflicts. To prevent these conflicts, the general approach is to sort the sub scalars first, and then assign different threads to handle sub scalars with different values. In fact, this process may be influenced by the distribution of scalars (secret inputs).

However, the most advantageous  $s$  for most implementations in this paper is approximately 8 (meaning that only  $t \cdot (2^8 - 1)$  bucket space is required). Therefore, our storage space permits us to assign independent bucket space to each thread. When we implement with the cuZK [LWY<sup>+</sup>23], we can also avoid write conflicts because cuZK itself can be considered as having resolved the issue of write conflicts by utilizing sparse matrix operations. Therefore, while the algorithm in [CPD<sup>+</sup>24] requires constant-time patches, the algorithms implemented in this paper do not.

### 3.2 From The General to The Specific

In the previous subsection, we have proposed our generic *elastic MSM* technique. To deepen our understanding of *elastic MSM*, we explain its relationship with two specific schemes in this subsection. Recall that with the scale of MSM  $n$ , the bit length of scalar  $\lambda$  and window size  $s$ , the Pippenger algorithm requires around  $\frac{\lambda}{s}(n + 2^{s+1})$  PADDs plus  $\lambda - s$  PDBLs. As we have discussed in **Subsection 3.1**, with  $\frac{\lambda}{s} = w \cdot k$ , our *elastic MSM* could be used to transform the scale of MSM from  $n$  to  $wn$ , the bit length of scalar from  $\lambda$  to  $ks$  but window size is still  $s$ , at the cost of preprocessing requiring  $n(w - 1)ks$  PDBLs and  $n(w - 1)$  extra memory space overhead. Then we could conclude that:

**Relationship with the *windowed MSM* technique in Pippenger algorithm.** It is obviously that if we set the parameter  $w = 1$  in our *elastic MSM*, then there is no preprocessing procedure at all. In this case, by the **Formula (4)**, it is clear that our *elastic MSM* degenerates into *windowed MSM* in Pippenger algorithm.

**Relationship with the *Computation Consolidation* technique in GZKP.** We find it intriguing that the *Computation Consolidation* technique, employed in GZKP [MXS<sup>+</sup>23], is encompassed as a specific instance within our *elastic MSM* framework. Turning to the other extreme, if we set the parameter  $k = 1$  in our *elastic MSM*, then  $w$  reaches its maximum value  $\frac{\lambda}{s}$ . The cost of preprocessing also reaches its maximum value  $n(\frac{\lambda}{s} - 1)s$  PDBLs, and  $n(\frac{\lambda}{s} - 1)$  extra memory space overhead. However, when *elastic MSM* with  $k = 1$  is implemented in the original Pippenger algorithm, there are only  $\frac{ks}{s}(wn + 2^{s+1}) = \frac{\lambda n}{s} + 2^{s+1}$  PADDs and  $\lambda - s = ks - s = 0$  PDBLs, which means that we have transferred many computation of PADDs to the preprocessing computation, and completely transferred the cost of PDBLs to the preprocessing computation. This transfer is exactly the same as the *Computation Consolidation* technique used in GZKP. Therefore, following our *elastic MSM*, we may be able to view the idea in *Computation Consolidation* from a higher dimension, and to some extent provide theoretical support for its optimality in cost of both Pippenger PADDs and PDBLs.

In addition to the extreme parameter selection, it is the parameter adjustment in other situations that is the core of our *elastic MSM*, and we will discuss it in detail in **Section 4**. In fact, the GZKP failed to view the preprocessing procedure from a higher perspective like we do, so the preprocessing MSM scheme in GZKP failed to adjust parameters itself, which leads to huge memory space overhead required to store the precomputed points. Though it has been mentioned in GZKP that adjustments of memory space overhead could be achieved through additional combination of a checkpoint strategy, this strategy incurs a significant amount of additional MSM computation.

## 4 The Elastic Pippenger Algorithm

For the reason that MSM is the most time-consuming operation in zkSNARK, it is very necessary for us to develop the efficient parallel MSM algorithm. And due to the outstanding performance of the Pippenger algorithm, parallel MSM methods based on it have been experimentally shown to perform better than the other MSM algorithms. Thus, in this section, as an example of analysis, we plug our *elastic MSM* technique into two commonly used parallel Pippenger algorithms and a more state-of-the-art MSM algorithm to get three new preprocessing parallel Pippenger-like algorithms—*elastic Pippengers*. Then we conduct a detailed theoretical comparison among the three *elastic Pippengers*, and the preprocessing parallel Pippenger-like algorithms composed of the GZKP's preprocessing scheme. Note that our *elastic MSM* technique is clearly modular and could be plugged in all Pippenger-like algorithms, which include *windowed MSM*, to flexibly optimize the original algorithm.

#### 4.1 Some Common Parallel Pippenger Algorithms

Based on the original Pippenger algorithm, it is easy to come up with three naive parallel Pippenger algorithms. While the recent work by [LWY<sup>+</sup>23] has summarized these approaches very clearly, for the completeness of our discussion, we also provide some explanations on these algorithms here.

**Approach 1.** The first naive approach can be regarded as a natural derivative of the original Pippenger algorithm. As deduced in the **Formula (1)**, we establish that  $\mathbf{Q} = \sum_{j=1}^{\frac{\lambda}{s}} 2^{(j-1)s} \mathbf{G}_j$ , signifying the existence of  $\frac{\lambda}{s}$  natural subtasks. Consequently, we could just arrange  $\frac{\lambda}{s}$  threads to perform these subtasks simultaneously. However, this approach provides a speedup of at most  $\frac{\lambda}{s}$ . This means that we cannot fully utilize the potential of GPUs for that the  $\frac{\lambda}{s}$  is usually much smaller than the number of cores provided by GPUs ( $\lambda$  typically ranges from 254 to 768, and  $s$  can be chosen at will).

**Complexity of the first naive approach.** Similar to the original Pippenger algorithm, the first naive approach initially converts the task using *windowed MSM* technique. Then, we only need to compute subtask results  $\mathbf{G}_j$ , for each  $j \in [1, \frac{\lambda}{s}]$  in parallel. These steps require, as analyzed earlier, at most  $n + 2^{s+1}$  PADDs. Finally, computing the MSM result with subtask results requires, as previously analyzed, at most  $\lambda - s$  PDBLs and  $\frac{\lambda}{s}$  PADDs. Thus, the entire first naive approach requires at most  $n + 2^{s+1} + \frac{\lambda}{s}$  PADDs and  $\lambda - s$  PDBLs.

**Approach 2.** The second naive approach is just a general method for dividing a large-scale task into small-scale modular subtasks. We use  $t$  to denote the number of total threads (w.l.o.g. we assume that  $n$  is divisible by  $t$ ), then we have  $\mathbf{Q} = \sum_{i=1}^n k_i \mathbf{P}_i = \sum_{j=1}^t \mathbf{Q}_j$ , where  $\mathbf{Q}_j = \sum_{i=1}^{\frac{n}{t}} k_{(j-1)\frac{n}{t}+i} \mathbf{P}_{(j-1)\frac{n}{t}+i}$  with  $j \in [1, t]$ . Thus, we can perform serial Pippenger algorithms in parallel for all the small-scale MSMs corresponding to  $\{\mathbf{Q}_j\}_{j \in [1, t]}$ . Finally, the result  $\mathbf{Q}$  can be computed using the parallel sum algorithm.

**Complexity of the second naive approach.** We can execute  $t$  Pippenger algorithms in parallel for all the small-scale MSM corresponding to  $\{\mathbf{Q}_j\}_{j \in [1, t]}$  with  $\lambda$ -bit scalars,  $s$ -bit window size and a MSM scale of  $\frac{n}{t}$ . This computation requires  $\frac{\lambda}{s} (\frac{n}{t} + 2^{s+1})$  PADDs and  $\lambda - s$  PDBLs. Additionally, the parallel sum algorithm contributes  $\log t$  PADDs. Thus, the entire second naive approach requires at most  $\frac{\lambda}{s} (\frac{n}{t} + 2^{s+1}) + \log t$  PADDs and  $\lambda - s$  PDBLs.

**Approach 3.** The third naive approach is a combination of the above two approaches. With a total of  $t$  threads, we can use the second approach followed by the first approach to decompose the original computation. Specifically, we denote the smallest subtask result as  $\mathbf{G}_{j,l}$ , then we have that

$$\mathbf{Q} = \sum_{i=1}^n k_i \mathbf{P}_i = \sum_{j=1}^{t/\frac{\lambda}{s}} \mathbf{Q}_j = \sum_{j=1}^{t/\frac{\lambda}{s}} \sum_{l=1}^{\frac{\lambda}{s}} 2^{(l-1)s} \mathbf{G}_{j,l} \quad (7)$$

**Complexity of the third naive approach.** According to the **Formula (7)**, we can compute our result  $\mathbf{Q}$  by first using the first approach to compute each of the small-scale MSM corresponding to  $\mathbf{Q}_{j \in [1, t/\frac{\lambda}{s}]}$  (with  $\lambda$ -bit scalars,  $s$ -bit window size and a MSM scale of  $\frac{n}{t/\frac{\lambda}{s}}$ ) in parallel. This step requires  $\frac{n}{t/\frac{\lambda}{s}} + 2^{s+1} + \frac{\lambda}{s} = \frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + \frac{\lambda}{s}$  PADDs and  $\lambda - s$  PDBLs. Then we only need to perform the parallel sum algorithm to sum all the  $\{\mathbf{Q}_j\}_{j \in [1, t/\frac{\lambda}{s}]}$ , which requires  $\log(t/\frac{\lambda}{s})$  PADDs. Thus, the entire third naive approach requires at most  $\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + \frac{\lambda}{s} + \log(t/\frac{\lambda}{s})$  PADDs and  $\lambda - s$  PDBLs.

It is clear that the third naive approach outperforms the other two approaches in theory when high parallelism is considered. However, it is worth noting that compared to the third approach, the second approach could be optimized more by preprocessing techniques. Therefore, the second approach is more worthy of preprocessing and often

used with preprocessing techniques. In fact, if combined with preprocessing techniques, the second approach may be as fast as the third approach in implementation. So in our next discussion, we will consider both of these approaches simultaneously. From now on, we call the second approach and the third approach just *fast parallel Pippenger 2* and *fast parallel Pippenger 3* respectively.

**MSM algorithm in cuZK.** Notably, the fastest parallel MSM algorithm recently proposed in [LWY<sup>+</sup>23] by Lu et al. has achieved nearly perfect linear speedup over the Pippenger algorithm [Pip76]. Its ideas mainly include the special parallel designing and the application of sparse matrices. From now on, we call the parallel MSM algorithm in [LWY<sup>+</sup>23] *fast parallel Pippenger 4*. For a more concise expression, we only use the conclusion of [LWY<sup>+</sup>23] in this paper, and we refer the reader to [LWY<sup>+</sup>23] for more details. Specifically, *fast parallel Pippenger 4* requires at most  $\frac{\lambda}{s}(\frac{n}{t} + \frac{2^{s+1}}{t}) + s + \log t$  PADDs plus  $\lambda$  PDBLs.

Next, we will plug our *elastic MSM* into *fast parallel Pippenger 2 (3, 4)* to obtain our *elastic Pippenger 2 (3, 4)*, and analyze them as an example. In fact, since our *elastic MSM* is a modular technique, it could also be plugged into any similar algorithm which has *windowed MSM* technique for speed improvement.

**MSM algorithm in [CPD<sup>+</sup>24].** Very recently, Chen et al. proposed several nice optimization techniques and introduced a novel GPU-accelerated MSM algorithm in the concurrent work [CPD<sup>+</sup>24]. Leveraging these techniques, [CPD<sup>+</sup>24] achieved significantly faster MSM computations on GPUs. Specifically, their implementation achieves 1.998 $\times$ , 1.821 $\times$ , and 1.818 $\times$  speedup on cuZK at scales of  $2^{21}$ ,  $2^{22}$  and  $2^{23}$ , respectively on an NVIDIA V100-SXM2 GPU card with BLS12-381. We are pleased to observe that our modular technique *elastic MSM* is also compatible with [CPD<sup>+</sup>24]. Notably, [CPD<sup>+</sup>24] incorporates optimizations for fundamental operations, such as point addition. These optimizations neither directly impact the computational complexity (where PADDs and PDBLs are considered to be basic operations), nor affect our estimation of the speedup. Consequently, we can estimate the speedup of [CPD<sup>+</sup>24] with *elastic MSM* in terms of computational complexity. We note that while the computational complexity of [CPD<sup>+</sup>24]’s scheme is better than that of the cuZK scheme, the difference is small. Therefore, it can be estimated that our technique could also bring substantial improvements to the scheme in [CPD<sup>+</sup>24] while flexibly controlling space overhead.

## 4.2 About the Checkpoint Strategy Used in GZKP

From our discussion in **Subsection 3.2**, as a special case of *elastic MSM* with  $k = 1$ , the *Computation Consolidation* technique proposed in GZKP [MXS<sup>+</sup>23] seems to be optimal in terms of MSM computation. However, as has also been mentioned in GZKP, the memory space overhead required to store the pre-computed points is over 5 GB at the MSM scale of  $2^{21}$ , and further grows as the MSM scale increases. Thus to endow the *Computation Consolidation* technique with the ability to adapt to different space limitations, GZKP also proposed a *Checkpoints* technique to better balance between the time and space saving. For the convenience of our subsequent comparison, we first introduce the *Checkpoints* technique in this subsection.

As we have discussed in **Subsection 3.2**, while GZKP could reduce the MSM computation, the cost of preprocessing of GZKP reaches its maximum value  $n(\frac{\lambda}{s} - 1)s$  PDBLs and  $n(\frac{\lambda}{s} - 1)$  extra memory space overhead. Fortunately, *Checkpoints* technique proposed in GZKP provides a solution. Instead of preprocessing and storing all points  $(2^s, 2^{2s}, \dots, 2^{(\frac{\lambda}{s}-1)s}) * \mathbf{P}_i$  for  $i \in [1, n]$ , we selectively preprocess and store only the points with fixed weights—these are called checkpoints. Specifically, we store points like  $(2^{M*s}, 2^{(2M)*s}, \dots, 2^{\lfloor \frac{\lambda/s}{M} \rfloor * M*s}) * \mathbf{P}_i$ , where  $1 \leq M < \frac{\lambda}{s}$  is an integer interval. Thus, when we sum all points in buckets, for each point, we could first locate the closest checkpoint to



the desired point, and then perform at most  $(M - 1)s$  PADDs to obtain the desired point. If  $t$  is the number of total threads, it is clear that the total additional preprocessing cost brought by the *Checkpoints* technique is at most  $n \left\lfloor \frac{\lambda/s}{M} \right\rfloor$  extra memory space overhead, and  $\frac{n}{t} \left\lfloor \frac{\lambda/s}{M} \right\rfloor Ms$  PDBLs. Moreover, the total additional MSM computation cost brought by *Checkpoints* technique is at most  $\frac{n}{t}(\frac{\lambda}{s} - 1)(M - 1)s$  PADDs. For the convenience of theoretical comparison, we summarize in **Table 1** the additional overhead brought by *Checkpoints* technique and our *elastic MSM* technique both in parallel setting.

**Table 1:** Additional preprocessing overhead comparison

	Space Overhead	Extra PDBLs (Prep)	Extra PADDs (MSM)
<i>Checkpoints</i> technique	$n \left\lfloor \frac{\lambda/s}{M} \right\rfloor$	$\frac{n}{t} \left\lfloor \frac{\lambda/s}{M} \right\rfloor Ms$	$\frac{n}{t}(\frac{\lambda}{s} - 1)(M - 1)s$
<i>elastic MSM</i> technique	$n(w - 1)$	$\frac{n}{t}(w - 1)ks$	0

This table shows us the extra overhead brought by *Checkpoints* technique and our *elastic MSM* technique both in parallel setting. We use (Prep) and (MSM) to represent the extra preprocessing overhead and the extra MSM computation overhead respectively.  $n$  is the MSM scale,  $t$  is the number of total threads,  $\lambda$  is the bit length of scalar,  $s$  is the window size, the flexible parameter  $M$  is an integer interval and the flexible parameter  $w, k$  satisfy  $\frac{\lambda}{s} = w \cdot k$ . The memory space overhead in theory is measured by the number of EC points that need to be additionally stored.

So far, it seems that while our *elastic MSM* technique could be used directly to balance between the time and space saving by adjusting the flexible parameter  $w, k$ , the *Computation Consolidation* technique and the *Checkpoints* technique both proposed in GZKP could also be used together to achieve the same thing by choosing appropriate  $M$ . However, we need to emphasize that in order to achieve the same goal (in other words, computing MSM under the same storage space limitations), the additional computational cost of our solution is much smaller than that of GZKP's solution as we will discuss next.

### 4.3 Theoretical Analysis and Comparison

From the discussion of the previous subsections, we could know that the preprocessing technique *Computation Consolidation* itself does not contain any flexible parameter, and when we plug it into the Pippenger-like parallel algorithms (we use *fast parallel Pippenger 2* (3, 4) as an example in this paper), it should be combined with the *Checkpoints* technique to support balance between the time and space saving. And we call *fast parallel Pippenger 2*, *fast parallel Pippenger 3* and *fast parallel Pippenger 4* combined with both the *Computation Consolidation* technique and *Checkpoints* technique *GZKP Pippenger 2*, *GZKP Pippenger 3* and *GZKP Pippenger 4* respectively. Recall that our preprocessing technique *elastic MSM* itself could be regarded as a modular and adaptive parameter configuration technique, so it is a plug-and-play technique, that is when we plug *elastic MSM* into the Pippenger-like parallel algorithms (*fast parallel Pippenger 2* (3, 4) as an example in this paper), we could directly obtain algorithms *elastic Pippenger 2*, *elastic Pippenger 3* and *elastic Pippenger 4*, which support balance between the time and space saving. In this subsection, we show the theoretical analysis about various complexities of *GZKP Pippenger 2* (3, 4) and *elastic Pippenger 2* (3, 4). Then, we evaluate and compare these algorithms in theory under some practical parameters.

**Algorithms with approach 2.** Recall that we have introduced in **Subsection 4.1**, to compute MSM, *fast parallel Pippenger 2* requires at most  $\frac{\lambda}{s}(\frac{n}{t} + 2^{s+1}) + \log t$  PADDs and  $\lambda - s$  PDBLs. As has been discussed in **Subsection 3.1**, to compute the MSM with  $n$  scale,  $\lambda$ -bit scalars and  $s$ -bit window size, our *elastic MSM* could be used to first transform this MSM instance into another with  $wn$  scale,  $ks$ -bit scalars and  $s$ -bit window size, where  $\frac{\lambda}{s} = w \cdot k$ . So the *elastic Pippenger 2* algorithm requires  $\frac{\lambda}{s} \cdot \frac{n}{t} + k \cdot 2^{s+1} + \log t$

PADDs and  $ks - s$  PDBLs to compute the MSM. From the **Subsection 3.2**, we could know that the *Computation Consolidation* technique is included as a special case in our *elastic MSM* ( $k = 1$ ). Thus, the *GZKP Pippenger 2* algorithm (regardless of *Checkpoints* technique) requires  $\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + \log t$  PADDs and 0 PDBLs to compute the MSM. Now, we summarize in **Table 2** the overall theoretical complexity of algorithms related to *fast parallel Pippenger 2*.

**Table 2:** Comparison of algorithms related to *fast parallel Pippenger 2*

	Space Overhead	Precomputing PDBLs	MSM PDBLs	MSM PADDs
<i>fast parallel Pippenger 2</i>	0	0	$\lambda - s$	$\frac{\lambda}{s} \cdot \frac{n}{t} + \frac{\lambda}{s} \cdot 2^{s+1} + \log t$
<i>GZKP Pippenger 2</i>	$n \left\lfloor \frac{\lambda/s}{M} \right\rfloor$	$\frac{n}{t} \left\lfloor \frac{\lambda/s}{M} \right\rfloor Ms$	0	$\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + \log t + \frac{n}{t} (\frac{\lambda}{s} - 1)(M - 1)s$
<i>elastic Pippenger 2</i>	$n(w - 1)$	$\frac{n}{t}(w - 1)ks$	$ks - s$	$\frac{\lambda}{s} \cdot \frac{n}{t} + k \cdot 2^{s+1} + \log t$

This table shows us the theoretical complexity of algorithms related to *fast parallel Pippenger 2*.  $n$  is the MSM scale,  $t$  is the number of total threads,  $\lambda$  is the bit length of scalar,  $s$  is the window size, the flexible parameter  $M$  is an integer interval and the flexible parameter  $w, k$  satisfy  $\frac{\lambda}{s} = w \cdot k$ . The memory space overhead in theory is measured by the number of EC points that need to be additionally stored.

Compared to *GZKP Pippenger 2*, it is easy for us to see from the **Table 2** that our *elastic Pippenger 2* sacrificed a very small amount of computational efficiency for MSM PDBLs and also a amount of computational efficiency for MSM PADDs (regardless the  $\frac{n}{t} (\frac{\lambda}{s} - 1)(M - 1)s$  MSM PADDs brought by the preprocessing of GZKP). However, these sacrifices are relatively small compared to the additional MSM PADDs overhead brought by GZKP preprocessing.

**Algorithms with approach 3.** Recall that we have introduced in **Subsection 4.1**, to compute MSM, *fast parallel Pippenger 3* requires at most  $\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + \frac{\lambda}{s} + \log(t/\frac{\lambda}{s})$  PADDs and  $\lambda - s$  PDBLs. As has been discussed in **Subsection 3.1**, to compute the MSM with  $n$  scale,  $\lambda$ -bit scalars and  $s$ -bit window size, our *elastic MSM* could be used to first transform this MSM instance into another with  $wn$  scale,  $ks$ -bit scalars and  $s$ -bit window size, where  $\frac{\lambda}{s} = w \cdot k$ . So the *elastic Pippenger 3* algorithm requires  $\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + k + \log(\frac{t}{k})$  PADDs and  $ks - s$  PDBLs to compute the MSM. From the **Subsection 3.2**, we could know that the *Computation Consolidation* technique is included as a special case in our *elastic MSM* ( $k = 1$ ). Thus, the *GZKP Pippenger 3* (regardless of *Checkpoints* technique) requires  $\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + 1 + \log t$  PADDs and 0 PDBLs to compute the MSM. **Table 3** shows the overall theoretical complexity of algorithms related to *fast parallel Pippenger 3*.

**Table 3:** Comparison of algorithms related to *fast parallel Pippenger 3*

	Space Overhead	Precomputing PDBLs	MSM PDBLs	MSM PADDs
<i>fast parallel Pippenger 3</i>	0	0	$\lambda - s$	$\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + \frac{\lambda}{s} + \log(t/\frac{\lambda}{s})$
<i>GZKP Pippenger 3</i>	$n \left\lfloor \frac{\lambda/s}{M} \right\rfloor$	$\frac{n}{t} \left\lfloor \frac{\lambda/s}{M} \right\rfloor Ms$	0	$\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + \log t + \frac{n}{t} (\frac{\lambda}{s} - 1)(M - 1)s$
<i>elastic Pippenger 3</i>	$n(w - 1)$	$\frac{n}{t}(w - 1)ks$	$ks - s$	$\frac{\lambda}{s} \cdot \frac{n}{t} + 2^{s+1} + k + \log(\frac{t}{k})$

This table shows us the theoretical complexity of algorithms related to *fast parallel Pippenger 3*.  $n$  is the MSM scale,  $t$  is the number of total threads,  $\lambda$  is the bit length of scalar,  $s$  is the window size, the flexible parameter  $M$  is an integer interval and the flexible parameter  $w, k$  satisfy  $\frac{\lambda}{s} = w \cdot k$ . The memory space overhead in theory is measured by the number of EC points that need to be additionally stored.

Compared to *GZKP Pippenger 3*, it is easy for us to see from the **Table 3** that we sacrificed a very small amount of computational efficiency for MSM PDBLs in exchange for a huge improvement in MSM PADDs efficiency. Moreover, it is clear that the overhead of MSM computation in *GZKP Pippenger 3* is the same as that in *GZKP Pippenger 2*. In fact, it is not difficult to conclude through simple reasoning that *GZKP Pippenger*

2 and *GZKP Pippenger 3* are actually the same algorithm. This also shows that the second approach could be optimized more by preprocessing techniques for that the third approach performs better than the second approach when only considering parallelism without preprocessing.

**Algorithms with MSM in cuZK.** Recall that we have introduced in **Subsection 4.1**, to compute MSM, *fast parallel Pippenger 4* requires at most  $\frac{\lambda}{s}(\frac{n}{t} + \frac{2^{s+1}}{t}) + s + \log t$  PADDs plus  $\lambda$  PDBLs. As has been discussed in **Subsection 3.1**, to compute the MSM with  $n$  scale,  $\lambda$ -bit scalars and  $s$ -bit window size, our *elastic MSM* could be used to first transform this MSM instance into another with  $wn$  scale,  $ks$ -bit scalars and  $s$ -bit window size, where  $\frac{\lambda}{s} = w \cdot k$ . So the *elastic Pippenger 4* algorithm requires  $\frac{\lambda}{s} \cdot \frac{n}{t} + k \cdot \frac{2^{s+1}}{t} + s + \log t$  PADDs and  $ks$  PDBLs to compute the MSM. From the **Subsection 3.2**, we could know that the *Computation Consolidation* technique is included as a special case in our *elastic MSM* ( $k = 1$ ). Thus, the *GZKP Pippenger 4* algorithm (regardless of *Checkpoints* technique) requires  $\frac{\lambda}{s} \cdot \frac{n}{t} + \frac{2^{s+1}}{t} + s + \log t$  PADDs and  $s$  PDBLs to compute the MSM. Now, we summarize in **Table 4** the overall theoretical complexity of algorithms related to *fast parallel Pippenger 4*.

**Table 4:** Comparison of algorithms related to *fast parallel Pippenger 4*

	Space Overhead	Precomputing PDBLs	MSM PDBLs	MSM PADDs
<i>fast parallel Pippenger 4</i>	0	0	$\lambda$	$\frac{\lambda}{s} \cdot \frac{n}{t} + \frac{\lambda}{s} \cdot \frac{2^{s+1}}{t} + s + \log t$
<i>GZKP Pippenger 4</i>	$n \left\lfloor \frac{\lambda/s}{M} \right\rfloor$	$\frac{n}{t} \left\lfloor \frac{\lambda/s}{M} \right\rfloor Ms$	$s$	$\frac{\lambda}{s} \cdot \frac{n}{t} + \frac{2^{s+1}}{t} + s + \log t + \frac{n}{t}(\frac{\lambda}{s} - 1)(M - 1)s$
<i>elastic Pippenger 4</i>	$n(w - 1)$	$\frac{n}{t}(w - 1)ks$	$ks$	$\frac{\lambda}{s} \cdot \frac{n}{t} + k \cdot \frac{2^{s+1}}{t} + s + \log t$

This table shows us the theoretical complexity of algorithms related to *fast parallel Pippenger 4*.  $n$  is the MSM scale,  $t$  is the number of total threads,  $\lambda$  is the bit length of scalar,  $s$  is the window size, the flexible parameter  $M$  is an integer interval and the flexible parameter  $w, k$  satisfy  $\frac{\lambda}{s} = w \cdot k$ . The memory space overhead in theory is measured by the number of EC points that need to be additionally stored.

Compared to *GZKP Pippenger 4*, it is easy for us to see from the **Table 4** that we sacrificed a very small amount of computational efficiency for MSM PDBLs in exchange for a huge improvement in MSM PADDs efficiency. Moreover, compared to *fast parallel Pippenger 4*, *elastic Pippenger 4* also has a good improvement in MSM computation. Notably, as discussed in [LWY<sup>+</sup>23], the optimal window size for *fast parallel Pippenger 4* is a little large under some practical parameters. That is, the  $s$  is often set to approximately 16 to maximize the advantages of *fast parallel Pippenger 4*. This means that when  $n$  is not particularly large, the term  $\frac{2^{s+1}}{t}$  has a high magnitude relative to the term  $\frac{\lambda}{s} \cdot \frac{n}{t}$ .

**Analysis.** We assume that the size of the storage space is only enough to accommodate  $Q$  extra points, according to the **Table 3** we have that  $n(w - 1) = Q$ , then  $w = \frac{Q}{n} + 1$ ,  $k = \frac{\lambda}{s(Q/n+1)}$ . Since  $w$  and  $k$  together represent the dimension of a matrix, both of them are at least equal to or greater than 1. Then we have  $\frac{\lambda}{s(Q/n+1)} \geq 1$ , that is  $Q \leq \frac{n\lambda}{s} - n$ . In fact, if the inequality  $Q \geq \frac{n\lambda}{s} - n$  holds, then there is enough space to store all the  $n(\frac{\lambda}{s} - 1)$  points, in which case our *elastic MSM* technique will degenerate into the *Computation Consolidation* technique. Now, we use some specific parameters to conduct a very rough theoretical evaluation for *GZKP Pippenger 2 (3)* and *elastic Pippenger 2 (3)* (we could also use the same method to theoretically evaluate *GZKP Pippenger 4* and *elastic Pippenger 4* to obtain conclusions similar to those in this subsection). If we set the parameter  $n = 2^{22}$ ,  $t = 2^{12}$ ,  $\lambda = 3 \cdot 2^8$ ,  $s = 2^3$ ,  $Q = 7 \cdot 2^{22}$ , then we have the following evaluation:

We first consider *fast parallel Pippenger 2* and *fast parallel Pippenger 3*. It is clear that the MSM computation in *fast parallel Pippenger 2* needs a total of  $3 \cdot 2^{15} + 3 \cdot 2^{14} + 12 \approx 4.5 \cdot 2^{15}$  PADDs and  $3 \cdot 2^8 - 2^3 \approx 3 \cdot 2^8$  PDBLs. The MSM computation in *fast parallel Pippenger 3* needs a total of  $3 \cdot 2^{15} + 2^9 + 3 \cdot 2^5 + 7 - \log 3 \approx 3 \cdot 2^{15}$  PADDs and  $3 \cdot 2^8 - 2^3 \approx 3 \cdot 2^8$

PDBLs.

Then we consider *GZKP Pippenger 2* and *GZKP Pippenger 3*, from  $n \left\lfloor \frac{\lambda/s}{M} \right\rfloor = Q$  we could know that the minimum  $M$  that satisfies this equation is 13. Then the check points are  $\{2^{13s}, 2^{26s}, 2^{39s}, 2^{52s}, 2^{65s}, 2^{78s}, 2^{91s}\} * \mathbf{P}_i$  for  $i \in [1, n]$ , the parallel pre-computation of which requires  $\frac{n}{t} \left\lfloor \frac{\lambda/s}{M} \right\rfloor Ms = 91 \cdot 2^{13}$  PDBLs. During the MSM computation, the points  $\{2^s, 2^{2s}, \dots, 2^{95s}\} * \mathbf{P}_i$  for  $i \in [1, n]$  need to be additionally computed from the check points by  $\frac{n}{t} \cdot 556s = 139 \cdot 2^{15}$  PADDs. So, it is clear that the MSM computation in *GZKP Pippenger 2* needs a total of  $139 \cdot 2^{15} + 3 \cdot 2^{15} + 2^9 + 12 \approx 142 \cdot 2^{15}$  PADDs. And the MSM computation in *GZKP Pippenger 3* needs a total of  $139 \cdot 2^{15} + 3 \cdot 2^{15} + 2^9 + 1 + 12 \approx 142 \cdot 2^{15}$  PADDs.

Finally, we consider *elastic Pippenger 2* and *elastic Pippenger 3*, from  $Q = 7 \cdot 2^{22}$  we know that  $w = \frac{Q}{n} + 1 = 8$ , then  $k = \frac{\lambda/s}{w} = 12$ . Thus, the preprocessing of *elastic Pippenger 2 (3)* requires  $\frac{n}{t}(w-1)ks = 21 \cdot 2^{15}$  PDBLs. While *elastic Pippenger 2 (3)* requires  $ks - s = 11 \cdot 2^3$  MSM PDBLs, the MSM PADDs required by *elastic Pippenger 2* and *elastic Pippenger 3* are just  $3 \cdot 2^{15} + 3 \cdot 2^{11} + 12 \approx 3 \cdot 2^{15}$  and  $3 \cdot 2^{15} + 2^9 + 12 + 10 - \log 3 \approx 3 \cdot 2^{15}$  respectively.

Using the same evaluation method as the above example, we compare the theoretical results of *fast parallel Pippenger 2 (3)*, *GZKP Pippenger 2 (3)* and *elastic Pippenger 2 (3)* under different storage space limitations. Due to our rough theoretical estimation, theoretical results about approach 2 and approach 3 are almost the same. Then, we present them in **Table 5**.

**Table 5:** Rough comparison of theoretical valuations

Extra Space Overhead	<i>fast parallel Pippenger 2 (3)</i>			<i>GZKP Pippenger 2 (3)</i>			<i>elastic Pippenger 2 (3)</i>		
	Precomputing PDBLs	MSM PDBLs	MSM PADDs	Precomputing PDBLs	MSM PDBLs	MSM PADDs	Precomputing PDBLs	MSM PDBLs	MSM PADDs
$7 \cdot 2^{22}$	0	$3 \cdot 2^8$	$4.5 (3) \cdot 2^{15}$	$22.75 \cdot 2^{15}$	0	$142 \cdot 2^{15}$	$21 \cdot 2^{15}$	$11 \cdot 2^3$	$3 \cdot 2^{15}$
$5 \cdot 2^{22}$	0	$3 \cdot 2^8$	$4.5 (3) \cdot 2^{15}$	$21.25 \cdot 2^{15}$	0	$186.75 \cdot 2^{15}$	$20 \cdot 2^{15}$	$15 \cdot 2^3$	$3 \cdot 2^{15}$
$3 \cdot 2^{22}$	0	$3 \cdot 2^8$	$4.5 (3) \cdot 2^{15}$	$18.75 \cdot 2^{15}$	0	$280.5 \cdot 2^{15}$	$18 \cdot 2^{15}$	$23 \cdot 2^3$	$3 \cdot 2^{15}$
$2 \cdot 2^{22}$	0	$3 \cdot 2^8$	$4.5 (3) \cdot 2^{15}$	$16.5 \cdot 2^{15}$	0	$373.25 \cdot 2^{15}$	$16 \cdot 2^{15}$	$31 \cdot 2^3$	$3 \cdot 2^{15}$
$2^{22}$	0	$3 \cdot 2^8$	$4.5 (3) \cdot 2^{15}$	$12.25 \cdot 2^{15}$	0	$579.5 \cdot 2^{15}$	$12 \cdot 2^{15}$	$47 \cdot 2^3$	$3 \cdot 2^{15}$

This table shows us the theoretical performance comparison under different preprocessing storage space limitations when we set the parameter  $n = 2^{22}$ ,  $t = 2^{12}$ ,  $\lambda = 3 \cdot 2^8$ ,  $s = 2^3$ .

We could draw a preliminary conclusion from the **Table 5** that the existing preprocessing method which supports a flexible balance between time and space (that is the preprocessing in GZKP), incurs a significant amount of additional MSM computation. Compared to *GZKP Pippenger 2 (3)*, it is obvious that our *elastic Pippenger 2 (3)* requires less PDBLs during the preprocessing. Moreover, our *elastic Pippenger 2 (3)* requires significantly less MSM PADDs while only increasing the very few extra MSM PDBLs cost. Notably, as a norm in Pippenger-like algorithms, if we assume for simplicity the computational cost of PDBLs and that of PADDs in EC are the same, it is clear that extra MSM PDBLs cost in our *elastic Pippenger* could be just ignored. Thus, when operating within the same space limitations, our preprocessing method exhibits faster preprocessing time and leads to more efficient MSM computation compared to the existing preprocessing method, which supports a flexible balance between time and space. Moreover, compared to *fast parallel Pippenger 2 (3)*, it is clear that when the storage space is limited, the additional MSM overhead caused by the preprocessing in GZKP is very high, to the extent that it is even inferior to schemes without preprocessing. However, our *elastic Pippenger 2 (3)* indeed enjoys the benefits of preprocessing even under varying storage space limitations. Thus, *elastic MSM* is the first preprocessing technique to retain the improved MSM computation brought by preprocessing under varying storage space limitations.

## 5 Evaluation

In this section, we first give our experimental setting in **Subsection 5.1**. Then, we show some experimental results of *fast parallel Pippenger 2 (3, 4)*, *GZKP Pippenger 2 (3)* and *elastic Pippenger 2 (3, 4)* in **Subsection 5.2** to highlight the improvement that our preprocessing parallel MSM algorithm provides.

### 5.1 Experimental Setup

**Methodology.** The experiments are conducted on a server with dual 2.20 GHz Inter Xeon E5-2630 processors and 128 GB DRAM, running Ubuntu 20.04.6 with CUDA Toolkit 11.6. It is equipped with one NVIDIA GeForce GTX 2080Ti GPU card (with 11 GB memory) and one NVIDIA Tesla M40 card (with 24 GB memory), the former is utilized for *fast parallel Pippenger 2 (3)*, *GZKP Pippenger 2 (3)* and *elastic Pippenger 2 (3)* while the latter for *fast parallel Pippenger 4* and *elastic Pippenger 4*. The CPU-GPU data transfer is completed through PCI-E. Recognize that distinctions in hardware resources may exert a substantial influence on the outcomes of comparisons. Consequently, to facilitate comparisons with a degree of fairness, we meticulously orchestrate the evaluation of GPU implementations within a uniform testbed environment. Notably, just moving EC points from host memory to device memory takes much time on data transfer. Thus we overlap the CPU-GPU data transfer and device computing based on the multi-streaming technique during our experiments.

### 5.2 Performance

**Performance on two common parallel Pippenger algorithms.** Here, we examine and compare the execution time of the parallel MSM algorithms *fast parallel Pippenger 2 (3)*, *GZKP Pippenger 2 (3)* and *elastic Pippenger 2 (3)*. Similar to the pre-existing GPU implementation of Groth’s zk-SNARK in the Mina library [Min19], we implemented the algorithms about two common parallel Pippenger algorithms using low-level subroutines of Mina. Thus, we evaluate all these algorithms on the dense synthetic data created by libsnark [lib14] with the 753-bit MNT4753 curve.

**Table 6** and **Table 7** provide the evaluation results on the precomputing time and MSM time of *fast parallel Pippenger 2 (3)*, *GZKP Pippenger 2 (3)* and *elastic Pippenger 2 (3)* across various preprocessing storage space limitations. Given a range of practical parameters, it can be concluded from these tables that our *elastic Pippenger 2 (3)* has a slight advantage in precomputing time compared to *GZKP Pippenger 2 (3)* across all preprocessing storage space limitations. Notably, the precomputing time of *GZKP Pippenger 2* or *elastic Pippenger 2* is the same as the precomputing time of *GZKP Pippenger 3* or *elastic Pippenger 3*. This is because the only difference lies in the use of different MSM algorithms, but the preprocessing methods are consistent.

Additionally, the MSM time of *GZKP Pippenger 2* is the same as that of *GZKP Pippenger 3*. This is because they are essentially the same algorithm. However, the MSM time of *elastic Pippenger 2* is obviously longer than that of *elastic Pippenger 3*, due to significant differences in computational complexity. When the storage space is limited to storing  $7 \cdot 2^{22} - 2^{22}$  extra points, our *elastic Pippenger 2* achieves about  $90 - 192 \times$  speedup versus *GZKP Pippenger 2* and our *elastic Pippenger 3* achieves about  $90 - 223 \times$  speedup versus *GZKP Pippenger 3*. Therefore, under the same storage space limitations, our solution has significant efficiency advantages. From another perspective, we can infer that under the same time constraints, our solution requires less storage space. Furthermore, the stricter the restrictions on storage space are, the more advantageous our constructions are.



Moreover, compared to *fast parallel Pippenger 2 (3)*, the MSM time of *GZKP Pippenger 2(3)* is clearly unacceptable across various storage space limitations. However, the *elastic Pippengers 2(3)* algorithms do benefit from the improvements brought by preprocessing. When the storage space is limited to storing  $7 \cdot 2^{22} - 2^{22}$  extra points, the *elastic Pippenger 2* achieves about  $1.56 - 1.90\times$  speedup versus *fast parallel Pippenger 2* and our *elastic Pippenger 3* achieves about  $1.02 - 1.08\times$  speedup versus *fast parallel Pippenger 3*. Thus, the *fast parallel Pippenger 2* could be optimized more by preprocessing techniques. It seems that when the scale of MSM is relatively large (e.g.  $n = 2^{22}$ ), the improvement brought by our preprocessing technique is relatively limited. However, it is worth noting that in many scenarios where zero knowledge proof is applied, the preprocessing is offline and one-shot, and many provers often need to generate many proofs. Therefore, limited improvements can also bring huge improvements to the entire system in large-scale application scenarios.

**Table 6:** Performance results (in seconds) about approach 2 across various storage space limitations.

Extra Space Overhead	<i>fast parallel Pippenger 2</i>		<i>GZKP Pippenger 2</i>		<i>elastic Pippenger 2</i>	
	Precomputing time	MSM time	Precomputing time	MSM time	Precomputing time	MSM time
$7 \cdot 2^{22}$	0	48.834(1.90 $\times$ )	151.377	2302.257(89.64 $\times$ )	138.68	25.682
$5 \cdot 2^{22}$	0	48.834(1.81 $\times$ )	141.833	2731.024(100.97 $\times$ )	132.528	27.048
$3 \cdot 2^{22}$	0	48.834(1.70 $\times$ )	124.545	4579.522(159.45 $\times$ )	120.369	28.72
$2 \cdot 2^{22}$	0	48.834(1.65 $\times$ )	110.301	5301.532(178.67 $\times$ )	106.122	29.673
$2^{22}$	0	48.834(1.56 $\times$ )	81.609	6004.887(192.48 $\times$ )	79.309	31.198

This table shows us the experimental performance comparison under different preprocessing storage space limitations when we set the parameter  $n = 2^{22}, t = 2^{12}, \lambda \approx 3 \cdot 2^8, s = 2^3$ .

**Table 7:** Performance results (in seconds) about approach 3 across various storage space limitations.

Extra Space Overhead	<i>fast parallel Pippenger 3</i>		<i>GZKP Pippenger 3</i>		<i>elastic Pippenger 3</i>	
	Precomputing time	MSM time	Precomputing time	MSM time	Precomputing time	MSM time
$7 \cdot 2^{22}$	0	27.522(1.08 $\times$ )	151.377	2302.257(90.08 $\times$ )	138.68	25.559
$5 \cdot 2^{22}$	0	27.522(1.06 $\times$ )	141.833	2731.024(105.01 $\times$ )	132.528	26.008
$3 \cdot 2^{22}$	0	27.522(1.05 $\times$ )	124.545	4579.522(174.14 $\times$ )	120.369	26.298
$2 \cdot 2^{22}$	0	27.522(1.03 $\times$ )	110.301	5301.532(198.86 $\times$ )	106.122	26.660
$2^{22}$	0	27.522(1.02 $\times$ )	81.609	6004.887(223.45 $\times$ )	79.309	26.873

This table shows us the experimental performance comparison under different preprocessing storage space limitations when we set the parameter  $n = 2^{22}, t = 2^{12}, \lambda \approx 3 \cdot 2^8, s = 2^3$ .

**Table 8** and **Table 9** give the precomputing time and MSM time of *fast parallel Pippenger 2 (3)*, *GZKP Pippenger 2 (3)* and *elastic Pippenger 2 (3)* across various MSM scales. The parameters  $t, \lambda$  and  $s$  selected in this experiment are consistent with the parameters used in the previous experiment. We chose a moderate storage space limit from the previous experiment, which means that we fix  $Q = 3 \cdot 2^c$ , integer  $c \in [16, 22]$  in this experiment (note that  $Q$  should change with  $n$  to be meaningful). It can be concluded from these tables that our *elastic Pippenger 2 (3)* has a slight advantage in precomputing time compared to *GZKP Pippenger 2 (3)* across almost all MSM scales. As we have discussed above, the precomputing time of *GZKP Pippenger 2* or *elastic Pippenger 2* is the same as the precomputing time of *GZKP Pippenger 3* or *elastic Pippenger 3*, the MSM time



of *GZKP Pippenger 2* is the same as that of *GZKP Pippenger 3*. And the MSM time of *elastic Pippenger 2* is obviously longer than that of *elastic Pippenger 3*.

Additionally, when the MSM scale is limited to  $2^{16} - 2^{22}$ , our *elastic Pippenger 2* achieves about  $16 - 159\times$  speedup versus *GZKP Pippenger 2* and our *elastic Pippenger 3* achieves about  $32 - 174\times$  speedup versus *GZKP Pippenger 3*. Therefore, the larger the scale of MSM is, the more advantageous our constructions are.

Moreover, compared to *fast parallel Pippenger 2 (3)*, the MSM time of *GZKP Pippenger 2(3)* is clearly unacceptable across various MSM scales. However, the *elastic Pippenger 2(3)* algorithms do benefit from the improvements brought by preprocessing. When the MSM scale is limited to  $2^{16} - 2^{22}$ , the *elastic Pippenger 2* achieves about  $1.70 - 2.58\times$  speedup versus *fast parallel Pippenger 2* and the *elastic Pippenger 3* achieves about  $1.05 - 1.39\times$  speedup versus *fast parallel Pippenger 3*. Thus, the *fast parallel Pippenger 2* could be optimized more by preprocessing techniques. In addition, the smaller the scale of MSM is, the more advantageous our constructions are.

**Table 8:** Performance Results (in seconds) about approach 2 across various MSM scales.

MSM Scale	<i>fast parallel Pippenger 2</i>		<i>GZKP Pippenger 2</i>		<i>elastic Pippenger 2</i>	
	Precomputing time	MSM time	Precomputing time	MSM time	Precomputing time	MSM time
$2^{16}$	0	11.036(2.58 $\times$ )	1.937	69.334(16.18 $\times$ )	1.875	4.284
$2^{17}$	0	13.288(2.34 $\times$ )	3.835	139.512(24.55 $\times$ )	3.678	5.683
$2^{18}$	0	16.829(2.27 $\times$ )	7.664	279.942(37.76 $\times$ )	7.365	7.413
$2^{19}$	0	22.530(2.46 $\times$ )	15.397	564.264(61.56 $\times$ )	14.676	9.166
$2^{20}$	0	29.585(2.47 $\times$ )	30.075	1135.314(94.73 $\times$ )	29.520	11.985
$2^{21}$	0	36.927(2.09 $\times$ )	62.074	2264.368(128.10 $\times$ )	59.834	17.676
$2^{22}$	0	48.834(1.70 $\times$ )	124.545	4579.522(159.45 $\times$ )	120.369	28.72

This table shows us the experimental performance comparison under different MSM scales and corresponding storage space limitations when we set the parameter  $t = 2^{12}$ ,  $\lambda \approx 3 \cdot 2^8$ ,  $s = 2^3$ .

**Table 9:** Performance Results (in seconds) about approach 3 across various MSM scales.

MSM Scale	<i>fast parallel Pippenger 3</i>		<i>GZKP Pippenger 3</i>		<i>elastic Pippenger 3</i>	
	Precomputing time	MSM time	Precomputing time	MSM time	Precomputing time	MSM time
$2^{16}$	0	2.973(1.39 $\times$ )	1.937	69.334(32.32 $\times$ )	1.875	2.145
$2^{17}$	0	4.774(1.17 $\times$ )	3.835	139.512(34.17 $\times$ )	3.678	4.083
$2^{18}$	0	7.188(1.11 $\times$ )	7.664	279.942(43.21 $\times$ )	7.365	6.479
$2^{19}$	0	8.937(1.09 $\times$ )	15.397	564.264(68.92 $\times$ )	14.676	8.187
$2^{20}$	0	11.315(1.07 $\times$ )	30.075	1135.314(107.46 $\times$ )	29.520	10.565
$2^{21}$	0	16.625(1.06 $\times$ )	62.074	2264.368(144.77 $\times$ )	59.834	15.641
$2^{22}$	0	27.522(1.05 $\times$ )	124.545	4579.522(174.14 $\times$ )	120.369	26.298

This table shows us the experimental performance comparison under different MSM scales and corresponding storage space limitations when we set the parameter  $t = 2^{12}$ ,  $\lambda \approx 3 \cdot 2^8$ ,  $s = 2^3$ .

**Performance on the MSM algorithm in cuZK.** Now, we test what improvements our technique can bring to the most state-of-the-art parallel MSM algorithm cuZK [LWY<sup>+</sup>23]. Notably, both the theoretical valuations in **Table 5** and the above performance results already indicated a fact that the GZKP preprocessing incurs unacceptable additional MSM computation when the preprocessing space is limited. Thus, we will only examine and compare the execution time of the parallel MSM algorithms *fast parallel Pippenger 4* and *elastic Pippenger 4* next. In fact, the original work by [LWY<sup>+</sup>23] did not incorporate

any preprocessing steps, and the authors expressed concerns about the storage overhead associated with pre-computation methods. However, *elastic MSM* is exactly a preprocessing technique with better time-space flexibility.

Here, we implement the MSM algorithm of cuZK based on the pre-existing cuZK library [cuZ23]. We also evaluate all these algorithms on the dense synthetic data created by libsnark [lib14] with the 753-bit MNT4753 curve. Notably, as discussed in [LWY<sup>+</sup>23], the optimal window size for *fast parallel Pippenger 4* is a little large under some practical parameters. That is, the  $s$  is often set to approximately 16 to fully leverage the advantages of *fast parallel Pippenger 4*. Thus, the parameter  $s$  in the upcoming experiment follows the recommended setting of [LWY<sup>+</sup>23], while keeping other parameters consistent with our other experiments.

**Table 10** provides the evaluation results on the precomputing time and MSM time of *fast parallel Pippenger 4* and *elastic Pippenger 4* across various preprocessing storage space limitations. Notably, *elastic Pippenger 4* with 0 extra space overhead is equivalent to *fast parallel Pippenger 4*. It can be concluded from **Table 10** that across various storage space limitations, our *elastic Pippenger 4* achieves approximately 1.22 – 1.36 $\times$  improvement compared to *fast parallel Pippenger 4* when we set  $n = 2^{22}$ ,  $t = 2^{12}$ ,  $\lambda \approx 3 \cdot 2^8$ ,  $s = 2^4$ .

**Table 10:** Performance results (in seconds) about MSM in cuZK across various storage space limitations.

<i>elastic Pippenger 4</i>		
Extra Space Overhead	Precomputing time	MSM time
$7 \cdot 2^{22}$	451.881	15.411(1.36 $\times$ )
$5 \cdot 2^{22}$	372.162	15.579(1.35 $\times$ )
$3 \cdot 2^{22}$	323.881	15.820(1.33 $\times$ )
$2 \cdot 2^{22}$	308.600	16.405(1.28 $\times$ )
$2^{22}$	298.493	17.171(1.22 $\times$ )
0	0	20.964

This table shows us the experimental performance comparison under different preprocessing storage space limitations when we set the parameter  $n = 2^{22}$ ,  $t = 2^{12}$ ,  $\lambda \approx 3 \cdot 2^8$ ,  $s = 2^4$ . Notably, *elastic Pippenger 4* with 0 extra space overhead is equivalent to *fast parallel Pippenger 4*.

**Table 11** gives the precomputing time and MSM time of *fast parallel Pippenger 4* and *elastic Pippenger 4* across various MSM scales. The parameters  $t$  and  $\lambda$  selected in this experiment are consistent with the parameters used in the previous experiment. And  $s$  is set to 16 (as recommended in [LWY<sup>+</sup>23]). We fix  $Q = 3 \cdot 2^c$ , *integer*  $c \in [16, 22]$  in this experiment. It can be concluded from **Table 11** that our *elastic Pippenger 4* achieves about 1.33 – 1.91 $\times$  speedup versus *fast parallel Pippenger 4* when the MSM scale is limited to  $2^{16} - 2^{22}$ . Notably, *elastic Pippenger 4* performs better in small-scale setting. This is because when  $n$  is not large, even relative to term  $\frac{\lambda}{s} \cdot \frac{n}{t}$ , term  $k \cdot \frac{2^{s+1}}{t}$  holds a significant proportion.

In addition, the above experiments indicate that applying our technique to faster MSM algorithms would not always result in a smaller speed improvement. Note that the *fast parallel Pippenger 4* is faster than the *fast parallel Pippenger 3*. However, our technique has brought greater improvements to the *fast parallel Pippenger 4*. Therefore, the extent of improvement our technique can bring depends on the specific complexity of the combined algorithm. Thus, we believe that the *elastic MSM* technique could also provide all subsequent Pippenger-like algorithms with the possibility of performing preprocessing to accelerate the computation while flexibly controlling space overhead.

**Table 11:** Performance Results (in seconds) about MSM in cuZK across various MSM scales.

MSM Scale	<i>fast parallel Pippenger 4</i>		<i>elastic Pippenger 4</i>	
	Precomputing time	MSM time	Precomputing time	MSM time
$2^{16}$	0	0.982(1.87×)	5.218	0.524
$2^{17}$	0	1.462(1.91×)	10.238	0.764
$2^{18}$	0	2.089(1.72×)	20.149	1.218
$2^{19}$	0	3.791(1.70×)	39.944	2.224
$2^{20}$	0	6.517(1.53×)	79.797	4.271
$2^{21}$	0	11.213(1.46×)	159.811	7.697
$2^{22}$	0	20.964(1.33×)	323.881	15.820

This table shows us the experimental performance comparison under different MSM scales and corresponding storage space limitations when we set the parameter  $t = 2^{12}$ ,  $\lambda \approx 3 \cdot 2^8$ ,  $s = 2^4$ .

## 6 Conclusion

**Summary.** In this work, we presented *elastic MSM*, a novel, modular and adaptive configuration technique for running MSM algorithms on GPUs. This technique could also be regarded as a flexible preprocessing technique for the Pippenger algorithm. Specifically, by transforming the scalar vectors into scalar matrices for processing, we could increase the scale of MSM (or preprocessing) and decrease the bit-width of the scalar (or online computation). This transformation enables *elastic MSM* to retain the improved MSM computation brought by preprocessing, even under varying storage space limitations. When operating within the same space constraints, the preprocessing method resulting from this transformation exhibits faster preprocessing time and leads to more efficient MSM computation compared to the existing method that supports a flexible balance between time and space. The fundamental advantage lies in the fact that the existing preprocessing method, which allows for such flexibility, incurs a significant amount of additional MSM computation, whereas *elastic MSM* effectively avoids these extra computations. Moreover, we compared and analyzed the theoretical complexity of our algorithms and the experimental results on GPUs.

**Future work.** Our technique is modular, and we emphasize the importance of space in this paper. However, we only explored our technique on some state-of-the-art parallel MSM algorithms. And our primary focus has been on the extra space overhead introduced by preprocessing here. In the future, we plan to explore more applications that can utilize our technique. Moreover, we will consider the space required for the MSM algorithm itself (including intermediate processes) as well, and try to make some optimizations.

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